Trento 29/10/2025

Tensors in Quantum Cryptography

Claudia De Lazzari - PhD

Who I am

- Bachelor and Master degree in Mathematics – University of Padova
- PhD in Mathematics – University of Trento + Q@TN
  - Alessandra Bernardi
  - Iacopo Carusotto
- Quantum Cryptographer – QTI s.r.l., Firenze

Summer 2019

Spring 2022

Autumn 2025

QTI – Quantum Telecommunications Italy

QTI is a company **founded in October 2020** as a spin-off of the Italian National Institute of Optics of the National Research Council (CNR).

First Italian **Quantum Key Distribution (QKD)** company, design, development and production of industrial grade systems for quantum networks.

2020
QTI is founded as spin-off of CNR-INO

2021
Telsy (TIM Group) invests in QTI company share

2022
Telsy increases its share reaching 49%

2024
Telsy increases its share reaching 80%

Today
QTI is a Telsy's subsidiary company

35 people
90% Technical Staff
84% under 37

Summary

- Why Quantum Key Distribution (QKD)
- Prepare and Measure BB84 protocol (Bennet, Brasserd 1984)
  - Description
  - Sketch of tensorial formalism
  - Eavesdropper action

Why QKD?

Key Distribution

GOAL : distribution of a **secure** key between parties:for encryption/decription

Key Distribution

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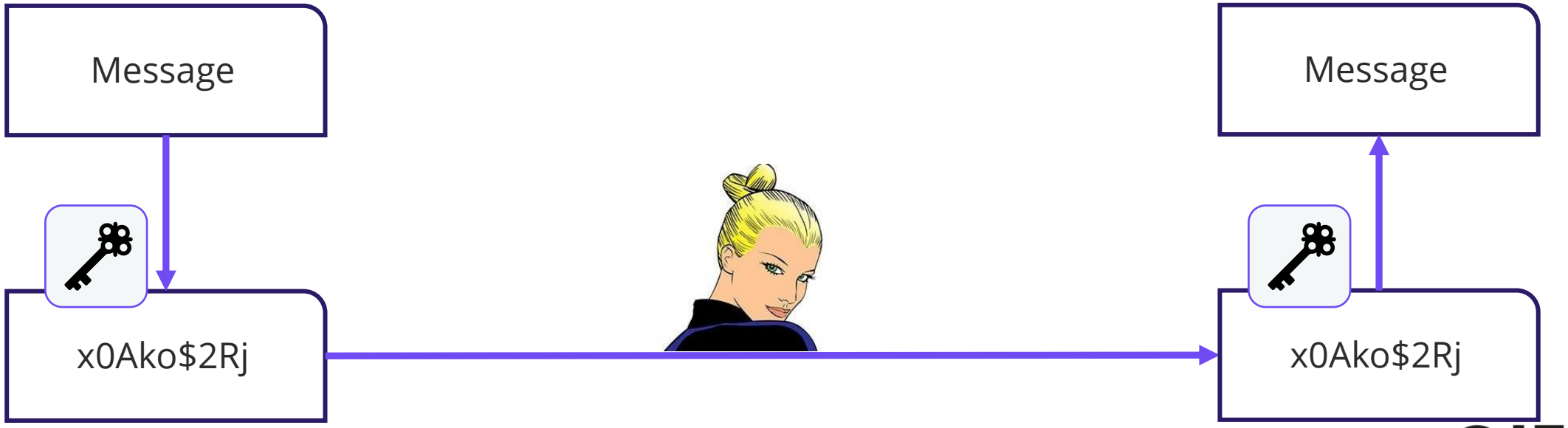
Message

Message

Key Distribution

GOAL : distribution of a **secure** key between parties:for encryption/decription



Quantum Key Distribution

GOAL : distribution of a key between parties **inconditionally** secure

All attacks, admitted by the laws of QM

Quantum Key Distribution

GOAL : distribution of a key between parties: **inconditionally** secure

All attacks, admitted by the laws of QM

Conventional key distribution

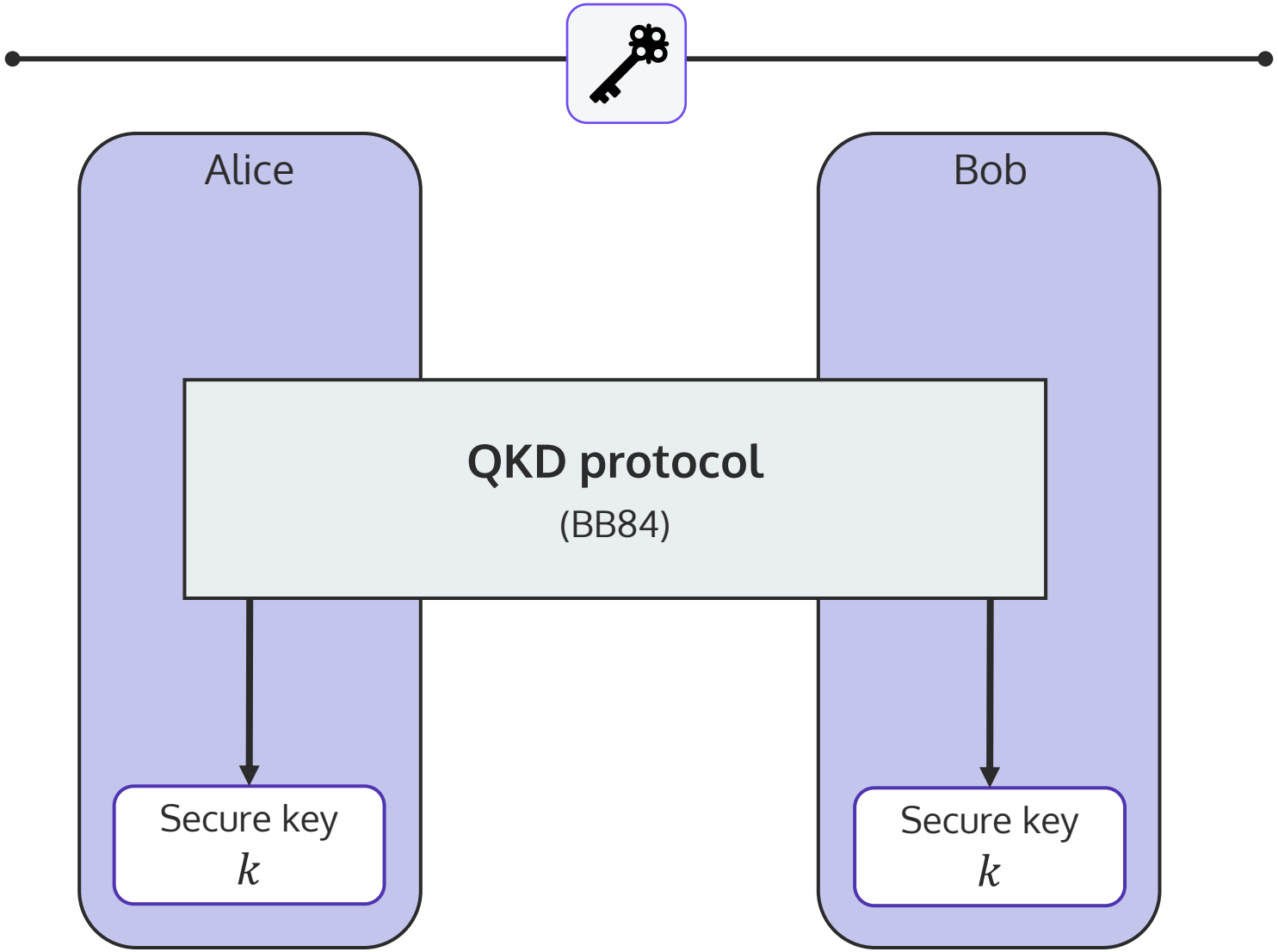
- Security relies on **hardness assumption**
- **Assumptions** on available technologies (computational power)
- **No** future-proof security

Quantum key distribution

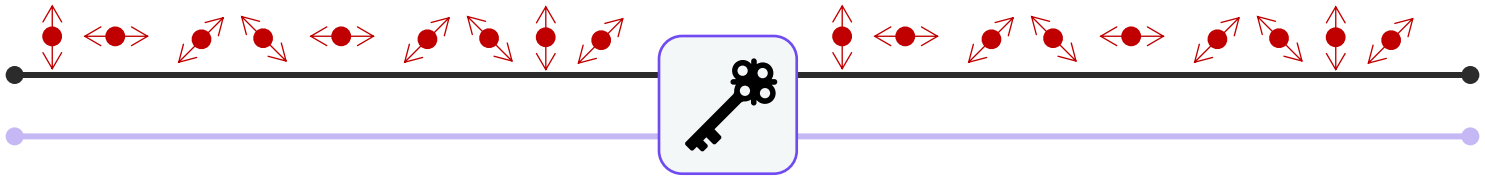
- **Security** mathematically quantified from physical laws
- No need to make **assumptions** on the enemy's resources
- **Future-proof** security

Prepare and Measure BB84

Quantum Key Distribution protocol



Quantum Channel



Alice

Bob

Quantum communication

Prep. and meas. of quantum states,
distributed in the quantum channel

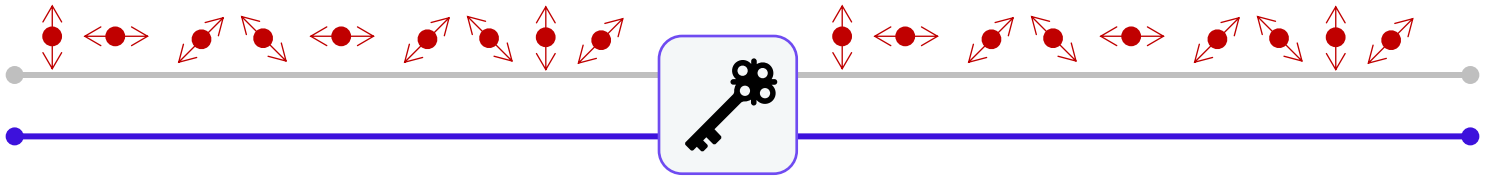
Classical post-processing

Elaboration of quantum results
via public discussion in the classical channel
(short-term authentication)

Secure key
k

Secure key
k

Classical Channel



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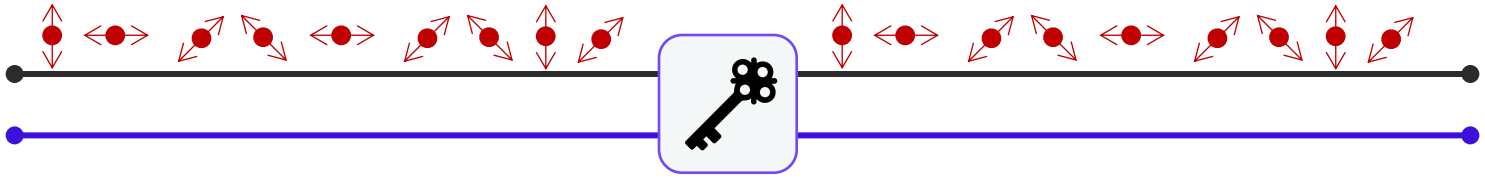
QKD Protocol



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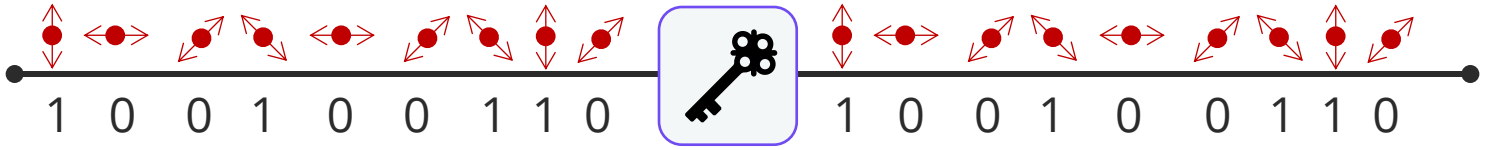
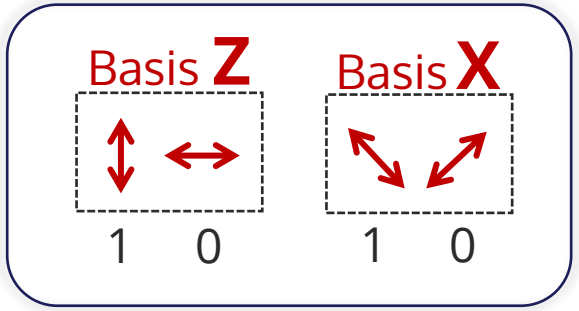
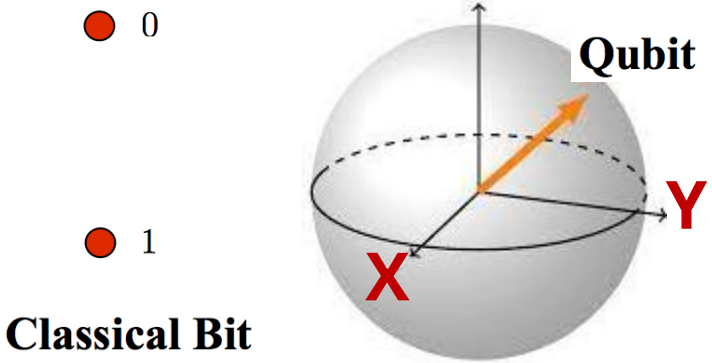




Quantum Superposition

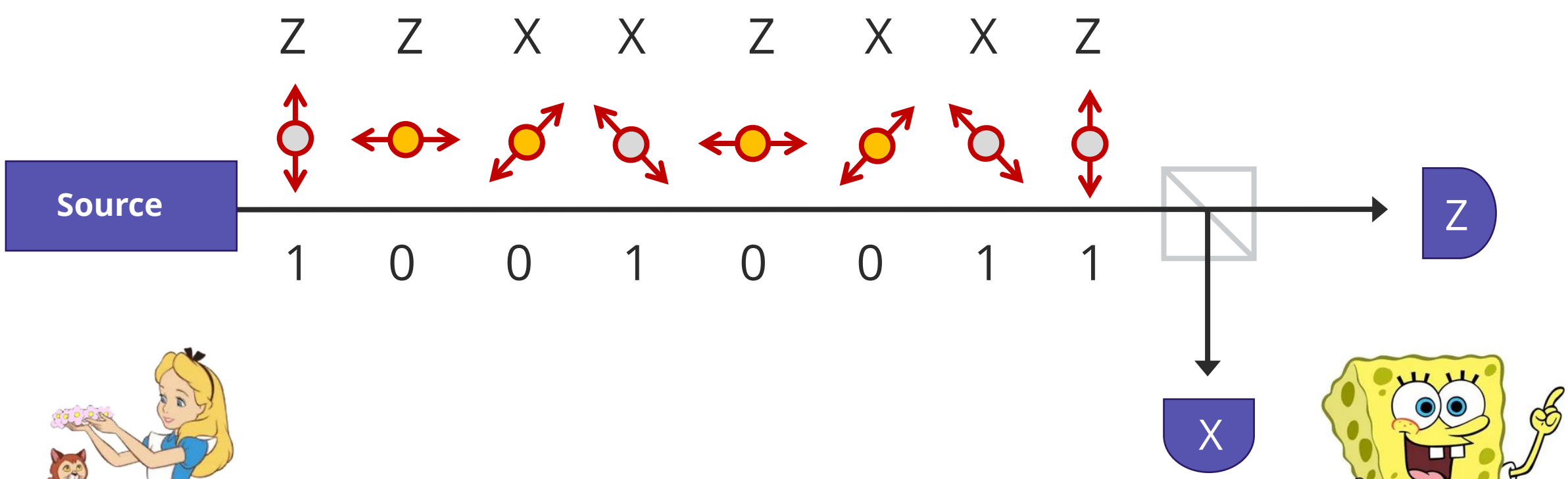
$$\leftrightarrow = \frac{\leftrightarrow + \updownarrow}{\sqrt{2}}$$

$$\nwarrow\!\!\searrow = \frac{\leftrightarrow - \updownarrow}{\sqrt{2}}$$



- Maximum uncertainty

Prepare and Measure



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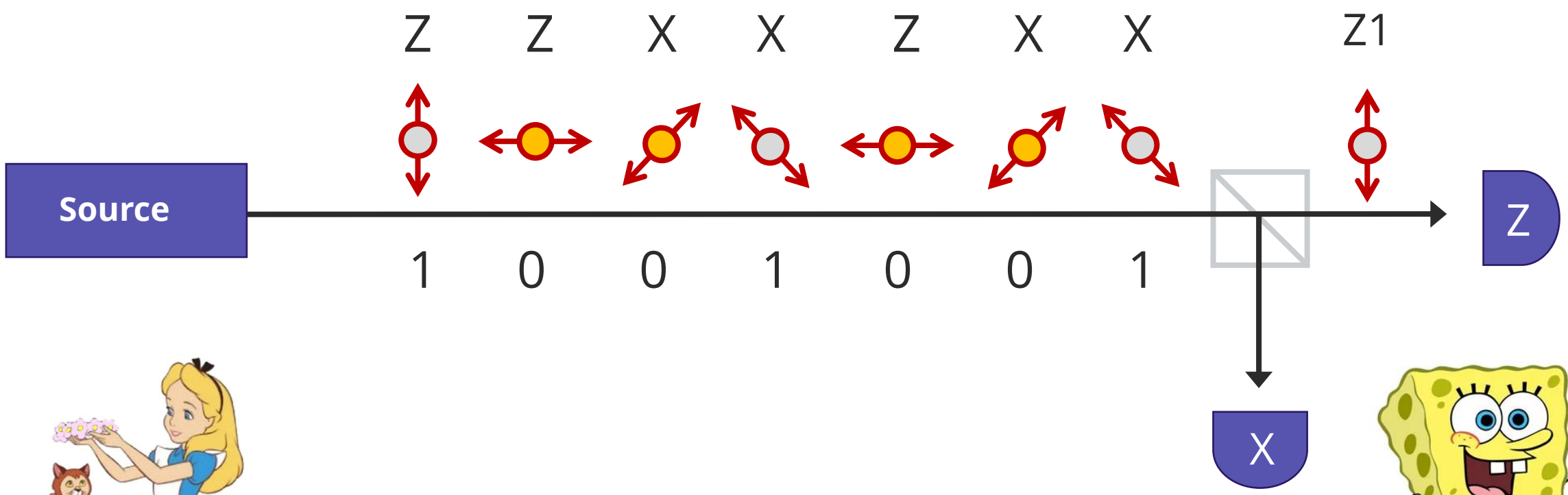

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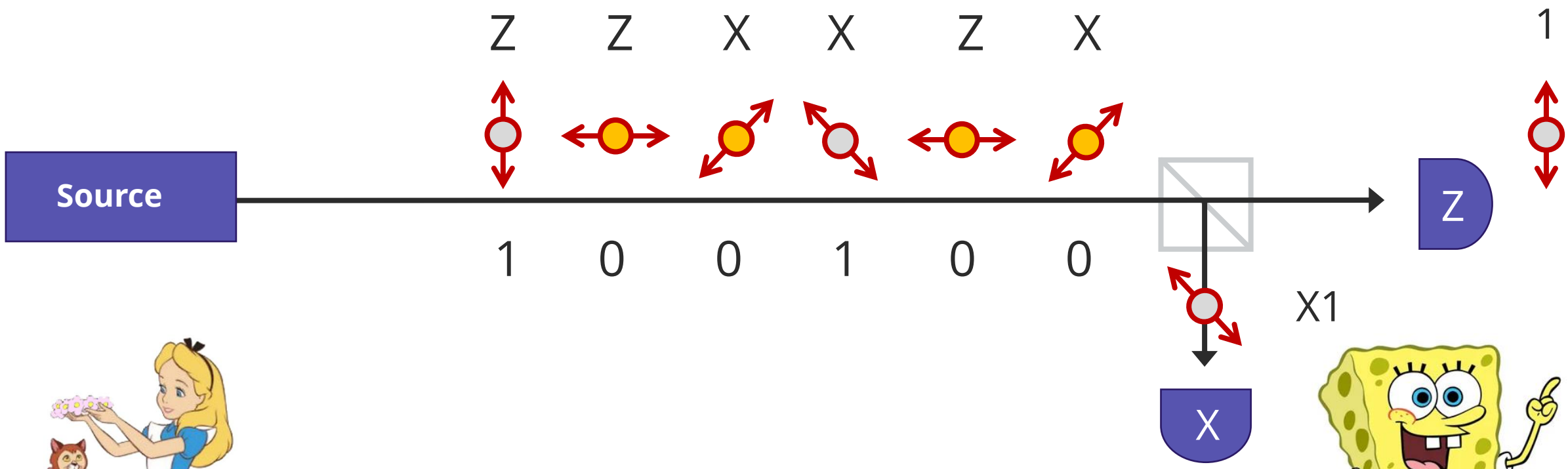

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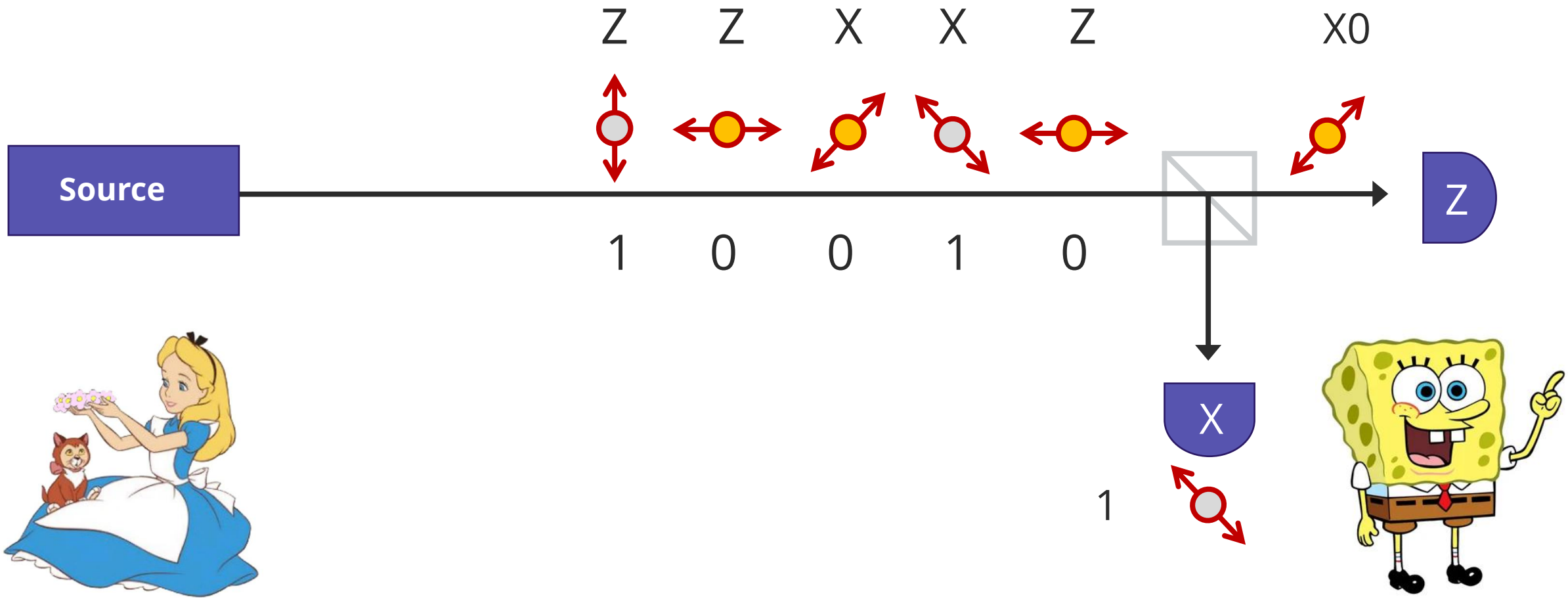

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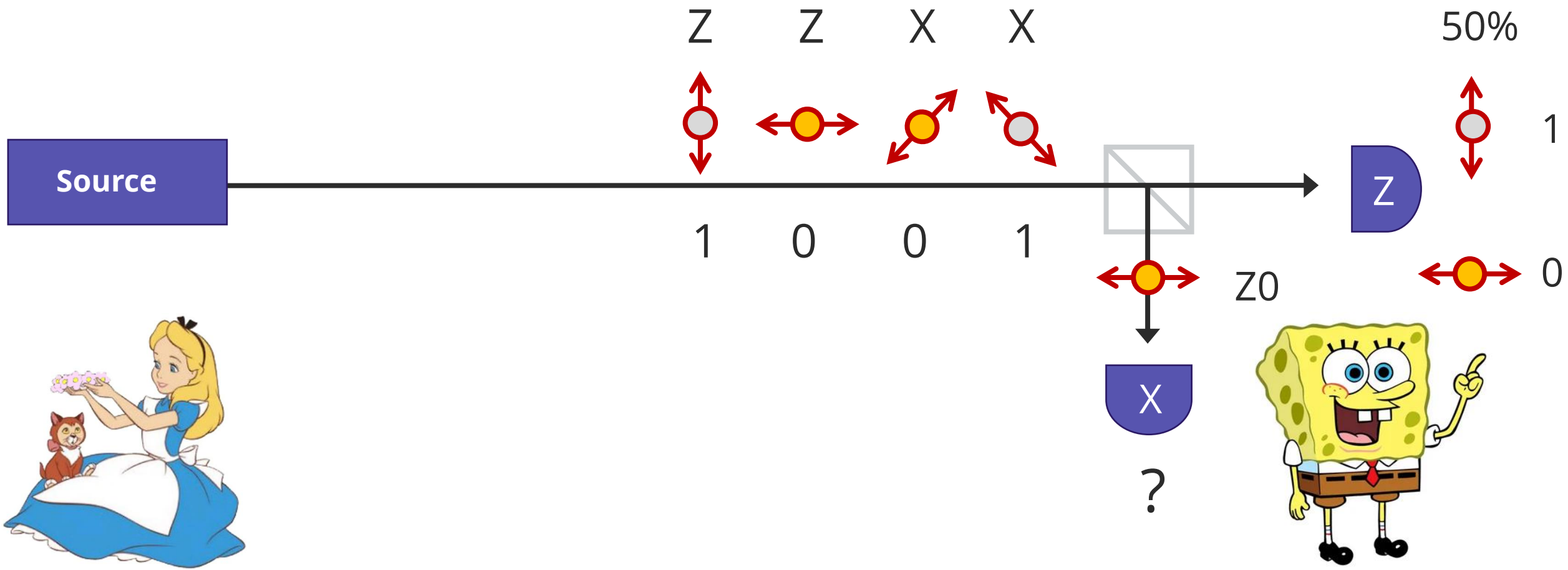

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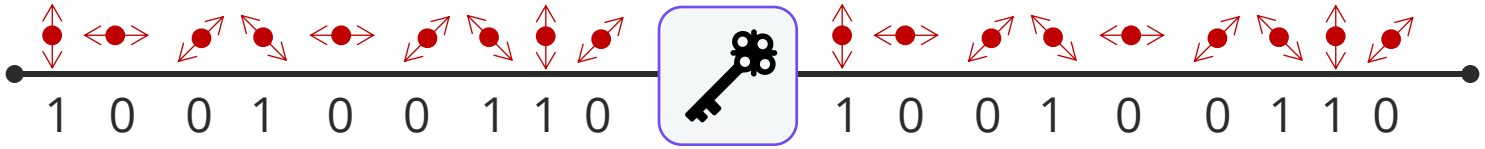
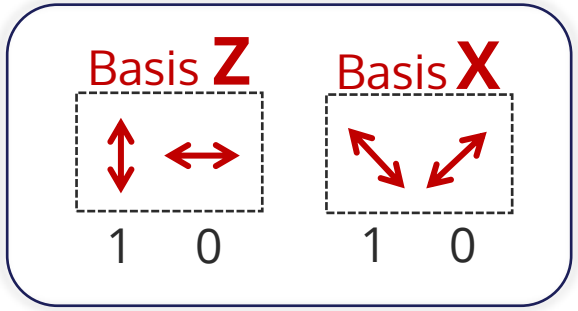
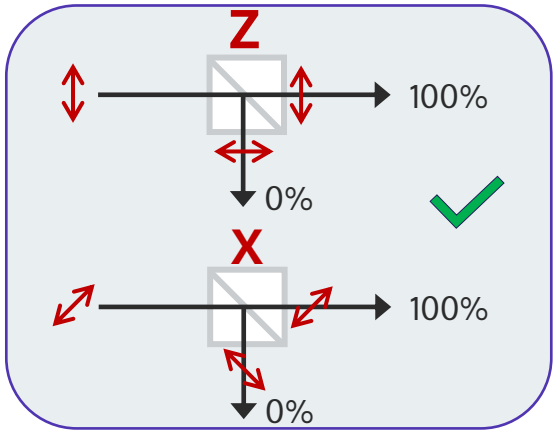
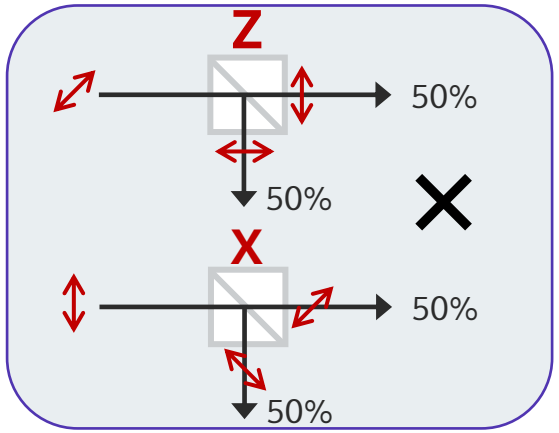


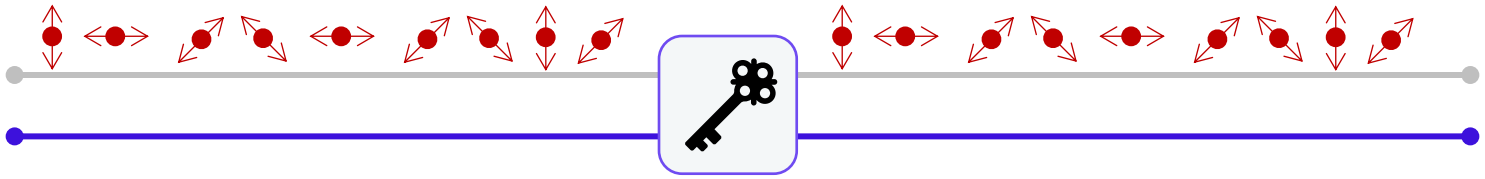


Quantum Measurements





Classical Channel



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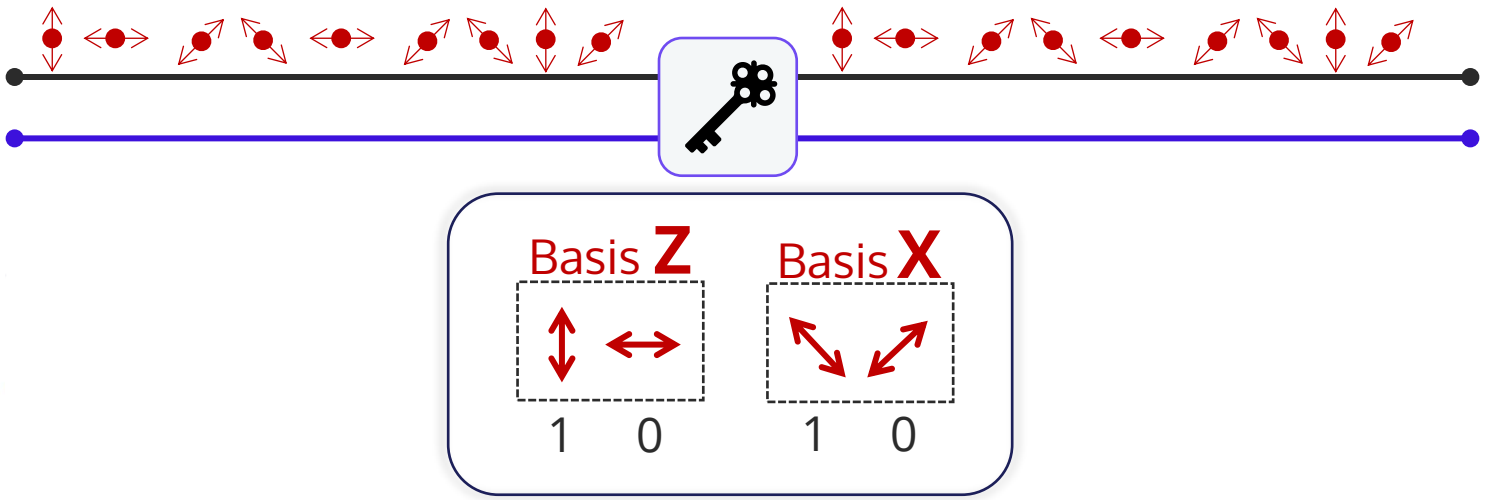
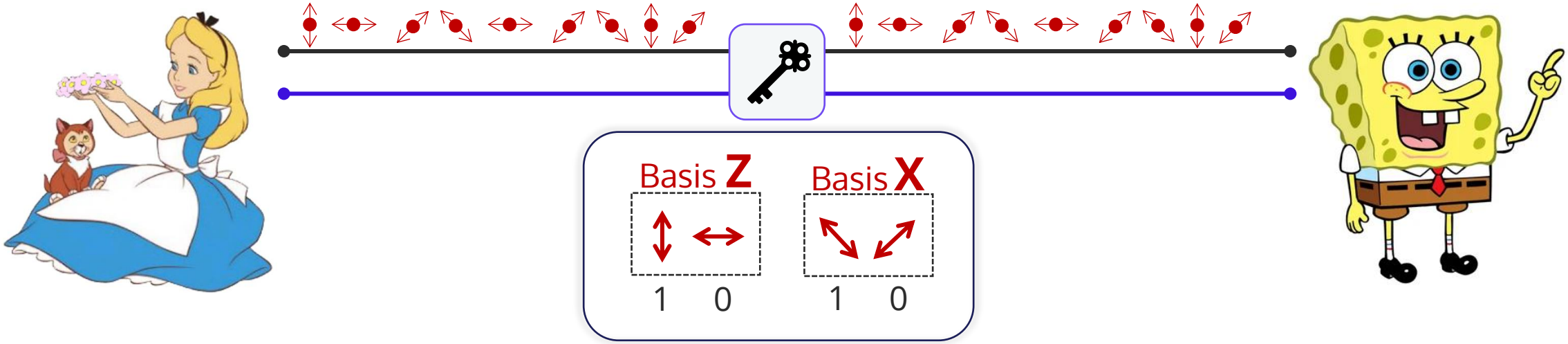
Sifting (bases riconciliation)



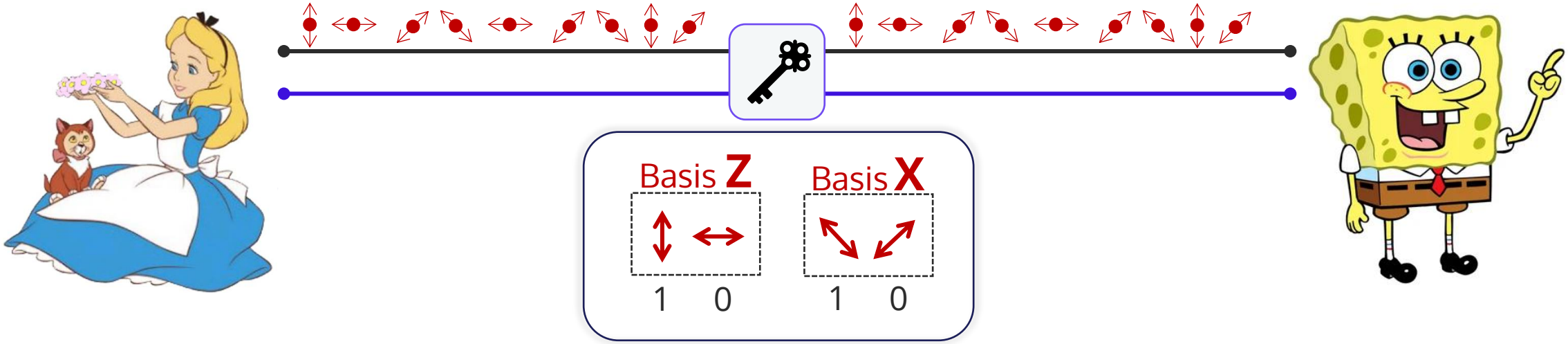
Alice	basis	Z	Z	X	X	Z	X	X	Z
	state	↕	↔	↗	↘	↔	↗	↘	↕
	bit	1	0	0	1	0	0	1	1
Bob	basis	Z	X	Z	X	X	X	Z	Z
	state	↕	↗	↕	↘	↘	↗	↕	↕
	bit	1	0	1	1	1	0	1	1

Sifting (bases riconciliation)



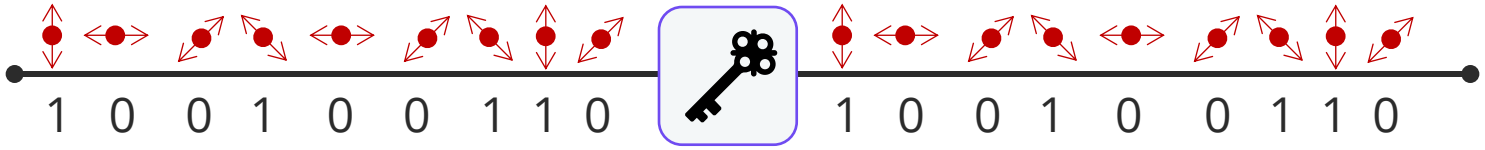
Alice	basis	Z	Z	X	X	Z	X	X	Z
	state	↕	↔	⤢	⤡	↔	⤢	⤡	↕
	bit	1	0	0	1	0	0	1	1
Bob	basis	Z	X	Z	X	X	X	Z	Z
	state	↕	⤢	↕	⤡	⤡	⤢	↕	↕
	bit	1	0	1	1	1	0	1	1
			✗✓	✗✗		✗✗		✗✓	

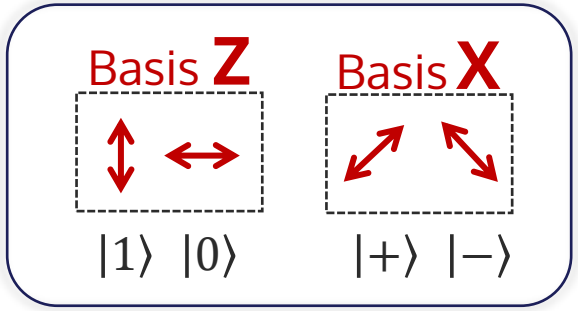
Sifting (bases riconciliation)



Alice	basis	**Z**	Z	X	**X**	Z	**X**	X	**Z**	public
	state	↕	↔	↗	↘	↔	↗	↘	↕	
	bit	1	0	0	1	0	0	1	1	
Bob	basis	**Z**	X	Z	**X**	X	**X**	Z	**Z**	public
	state	↕	↗	↕	↘	↘	↗	↕	↕	
	bit	1	0	1	1	1	0	1	1	
Sifted key		**1**	**--**	**--**	**1**	**--**	**0**	**--**	**1**	

Quantum communication





Quantum Superposition

$$\nearrow\!\!\searrow = \frac{\leftrightarrow + \updownarrow}{\sqrt{2}}$$

$$\nwarrow\!\!\searrow = \frac{\leftrightarrow - \updownarrow}{\sqrt{2}}$$

Bra-ket notation

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Max unbiased basis

$$\Pr(out_j) = |\langle j|i\rangle|^2 = \frac{1}{2}$$

$$j \in \{+,-\}, i \in \{0,1\}$$

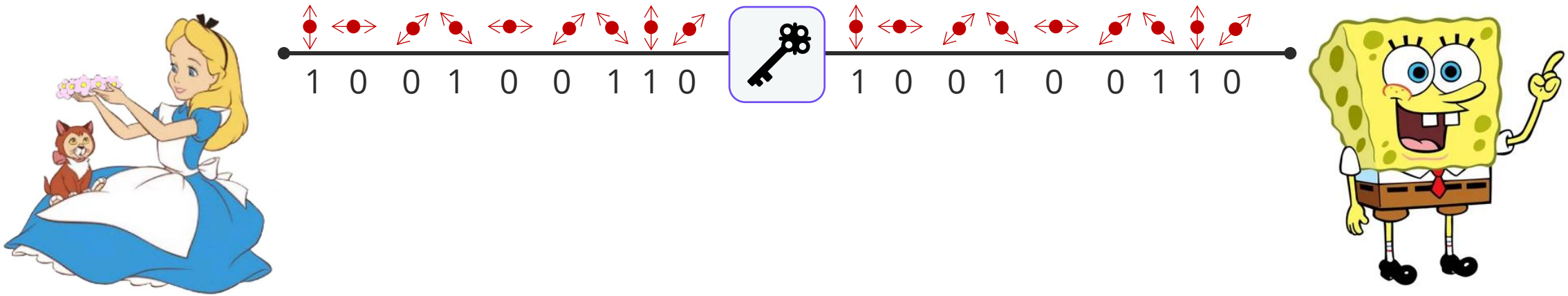
Tensor Spaces

State: $|1\rangle_A \in H_A$

Superposition: $|+\rangle_A = \frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}} \in H_A$

(Separable) composite state: $|k_2\rangle_A = \updownarrow \otimes \updownarrow = |1\rangle_A \otimes |1\rangle_A = |11\rangle_A \in H_A \otimes H_A$

Quantum communication



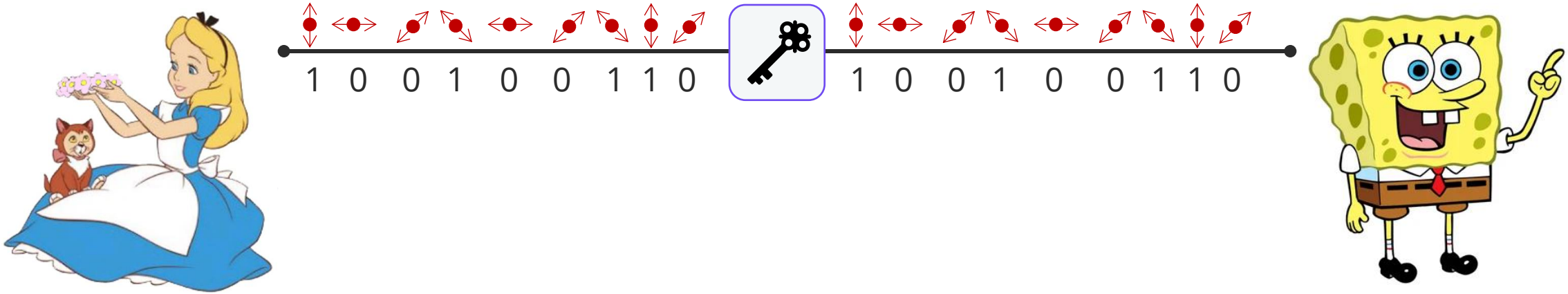
Alice state: $|1\rangle_A \otimes |0\rangle_A \otimes |+\rangle_A \otimes |-\rangle_A \otimes |0\rangle_A \otimes |+\rangle_A \otimes |-\rangle_A \otimes |1\rangle_A$

Classical bits: $k_A = (1,0,0,1,0,0,1,1)$

Bob state: $|1\rangle_B \otimes |+\rangle_B \otimes |1\rangle_B \otimes |-\rangle_B \otimes |-\rangle_B \otimes |+\rangle_B \otimes |1\rangle_B \otimes |1\rangle_B$

Classical bits: $k_B = (1,0,1,1,1,0,1,1)$

Quantum communication



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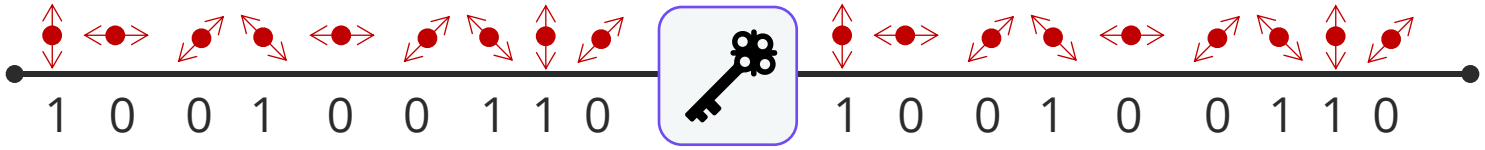
Quantum communication + Sifting



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Classical bits: $k_A = (1,0,1,1)$

Bob state: $|1\rangle_B \otimes |+\rangle_B \otimes |1\rangle_B \otimes |-\rangle_B \otimes |-\rangle_B \otimes |+\rangle_B \otimes |1\rangle_B \otimes |1\rangle_B$

Classical bits: $k_B = (1,1,0,1)$

After sifting $k_A = k_B = (1,1,0,1)$

Density matrix notation

Let $|\phi\rangle \in H$.

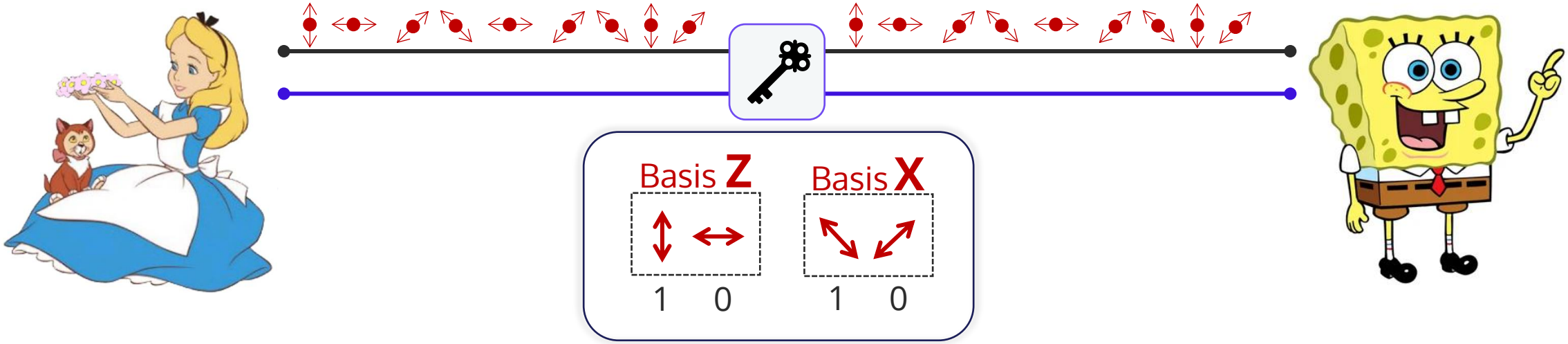
Bra notation: $\langle\phi| = |\phi\rangle^{\dagger}$, (transpose-conjugate).
Then $|\phi\rangle \otimes \langle\phi| =: |\phi\rangle\langle\phi| \in H \otimes H^{*}$ is an operator, called density matrix.

- A state is pure if its density matrix has the form $\rho = |\phi\rangle\langle\phi|$
- Otherwise, it is a (classical) statistical mixture

$$\rho = \sum_i \alpha_i |\phi_i\rangle\langle\phi_i|, \quad \text{with } \sum_i \alpha_i = 1.$$

The density matrix formalism gives a (general) description of the state.

Sifting (bases riconciliation)



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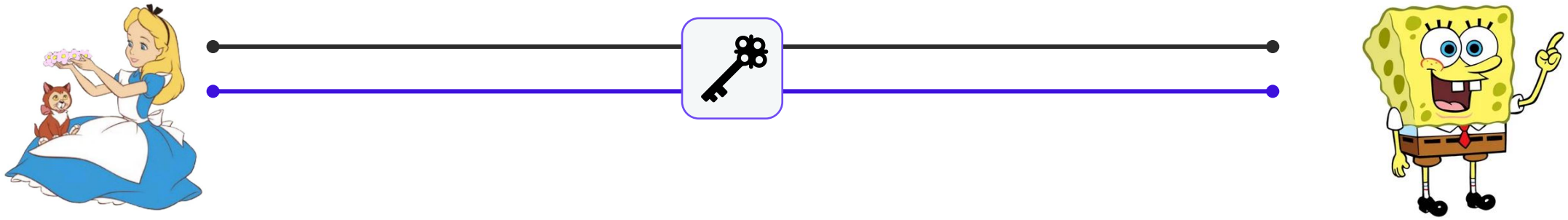
	state	↕	↔	↗	↘	↔	↗	↘	↕	
	bit	1	0	0	1	0	0	1	1	
Bob	basis	**Z**	X	Z	**X**	X	**X**	Z	**Z**	public
	state	↕	↗	↕	↘	↘	↗	↕	↕	
	bit	1	0	1	1	1	0	1	1	
Sifted key		**1**	**--**	**--**	**1**	**--**	**0**	**--**	**1**	

They expect

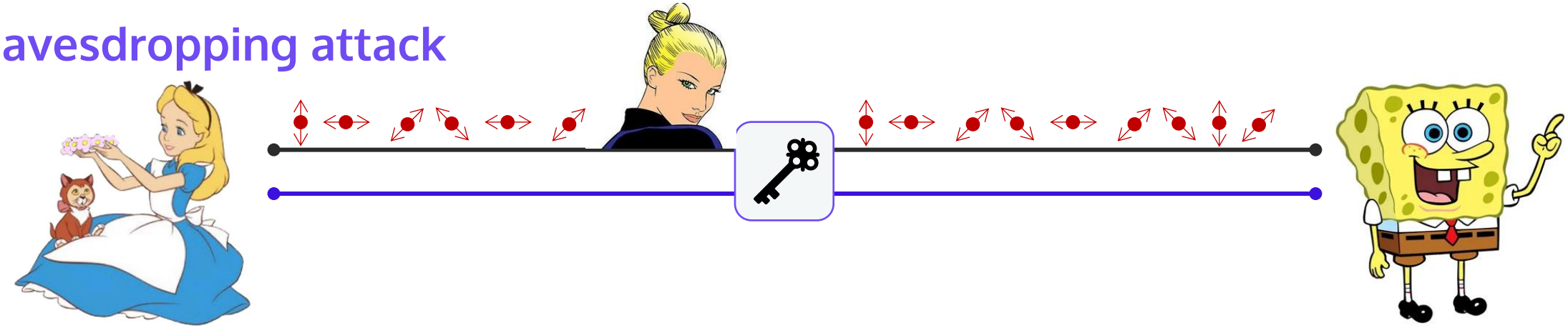


Let $K_A = K_B = K = \{0,1\}^N$ be the space of binary key of length N, $|K| = 2^N$.
Alice and Bob expect a (copy of a) shared key

$$\rho_{A,B}^{N,ideal} := \frac{1}{|K|} \sum_{k \in K} |k\rangle_{K_A}\langle k| \otimes |k\rangle_{K_B}\langle k|$$

→ the maximally correlated keys.
(Remember e.g. $|k\rangle = |010001101010\rangle$ of length N)

Eavesdropping attack



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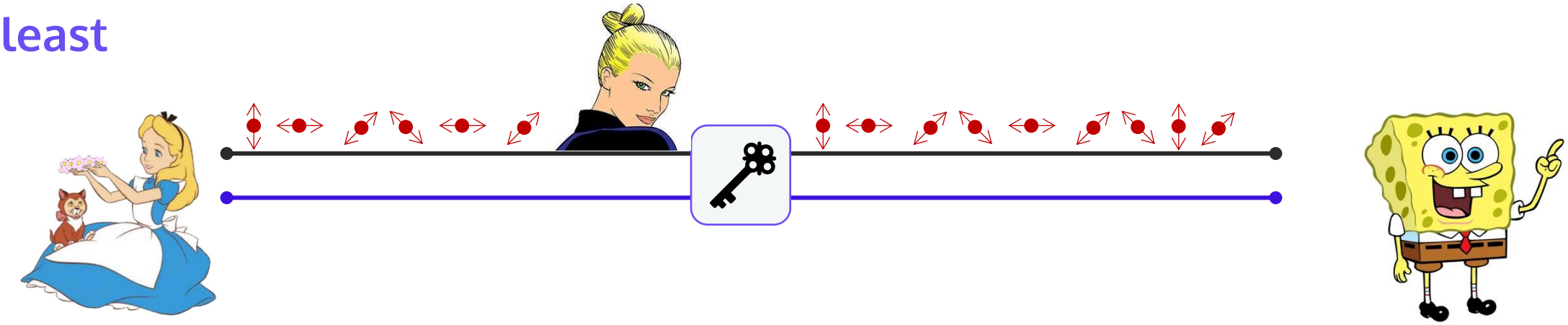
	state	↕	↔	↗	↘	↔	↗	↘	↕
	bit	1	0	0	1	0	0	1	1
Eve	basis	Z	Z	Z	Z	X	X	X	X
	state	↕	↔	↕	↔	↘	↗	↘	↗
	bit	1	0	1	0	1	0	1	0
Bob	basis	Z	X	Z	X	X	X	Z	Z
	state	↕	↗	↕	↗	↘	↗	↕	↕
	bit	1	0	1	0	1	0	1	1
Sifted key		**1**	**--**	**--**	**Error**	**--**	**0**	**--**	**1**

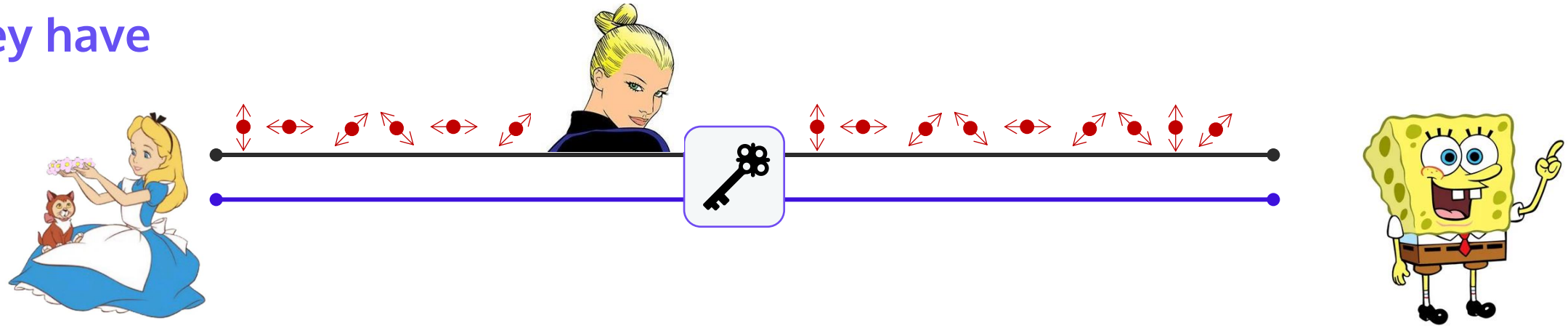
At least



At least

$$\rho_{A,B}^{N,ideal} := \frac{1}{|K|}\sum_{k\in K}(|k\rangle_A\langle k| \otimes |k\rangle_B\langle k|) \otimes \rho_E$$

They have



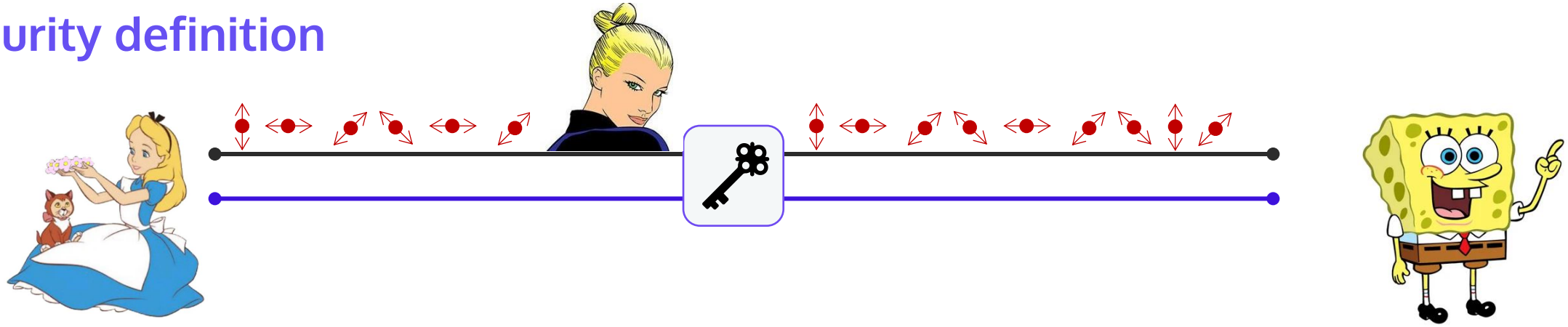
Let $K_A = K_B = K = \{0,1\}^N$ be the space of binary key of length N, $|K| = 2^N$.
Alice and Bob (and Eve) have the state of the form:

$$\rho^N_{A,B,E} = \sum_{k_A,k_B \in K} P_{k_A,k_B} |k_A\rangle_{K_A}\langle k_A| \otimes |k_B\rangle_{K_B}\langle k_B| \otimes \rho_E^{(k_A,k_B)}$$

with $\rho_E^{(k_A,k_B)} \in H_E$, Eve's final conditioned side-information.

In general: $\neq \rho^{N,ideal}_{A,B} = \frac{1}{|K|}\sum_{k\in K}(|k\rangle_A\langle k| \otimes |k\rangle_B\langle k|) \otimes \rho_E$

Security definition



At the end of a QKD protocol, Alice and Bob hold the final keys, while Eve has a final side-information. Let $\epsilon_{sec} \in [0,1]$.
We say the protocol is ϵ_{sec}-secure if

$$d\left(\rho_{A,B,E}^{\ell}, \rho_{A,B}^{\ell,ideal} \otimes \rho_E\right) \leq \epsilon_{sec}$$

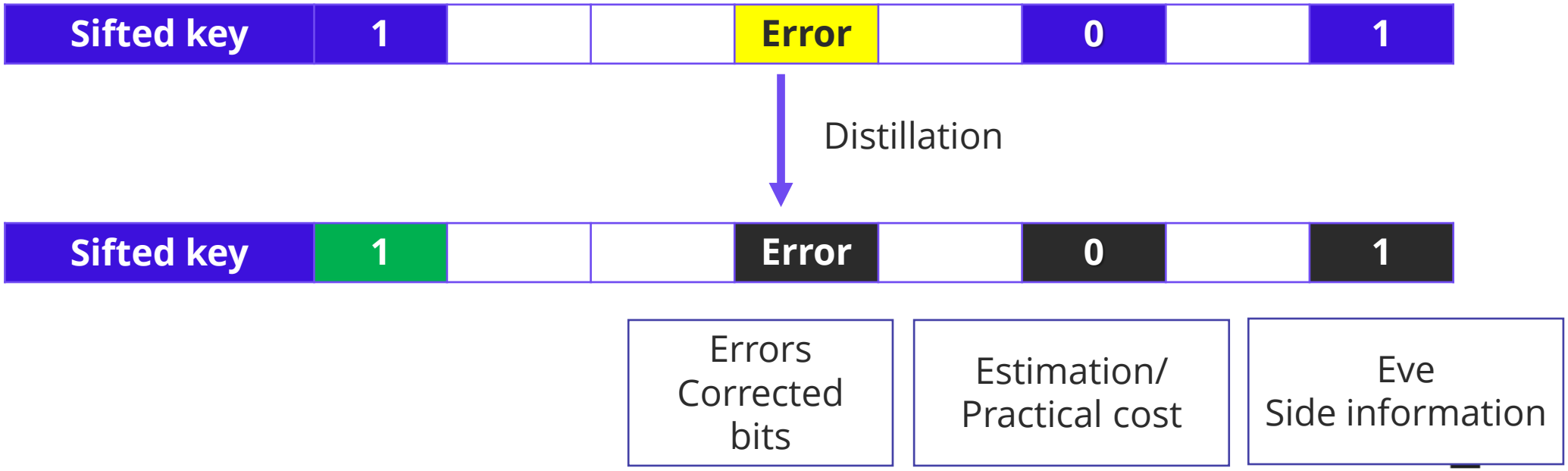
→ Quantification of distinguishability

Main considerations

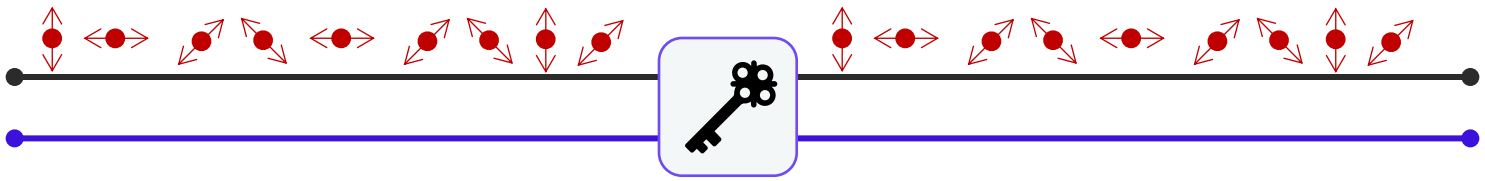
The ***tensor*** $\rho_{A,B,E}$ is the real (not ideal) state:

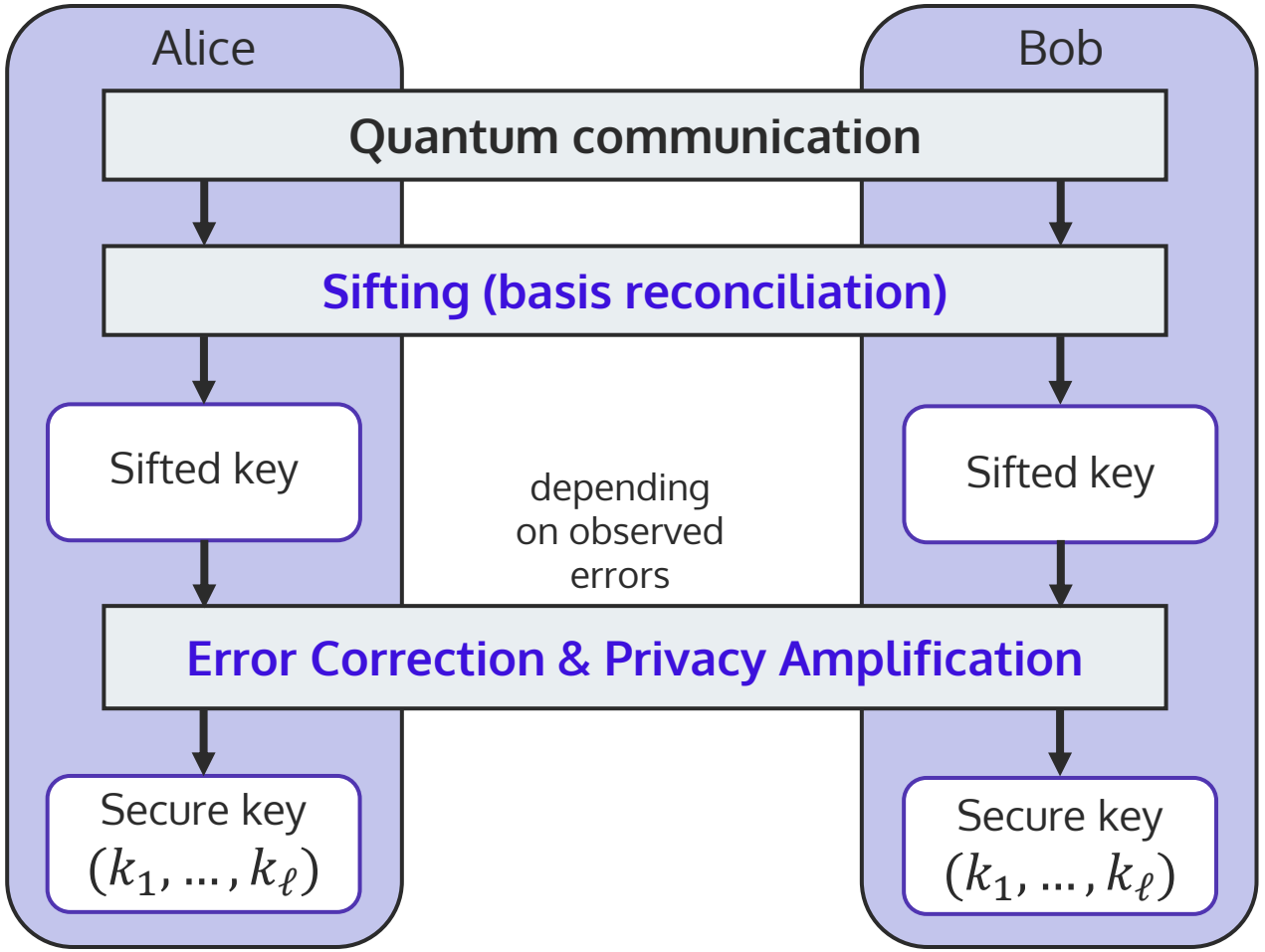
- It includes modelled attacks (if specifically assumed) + *setup imperfections*

- From the model, you must
  - *Find* errors → correct them
  - *Estimate* (bounds on) Eve's information → privacy amplify to ℓ

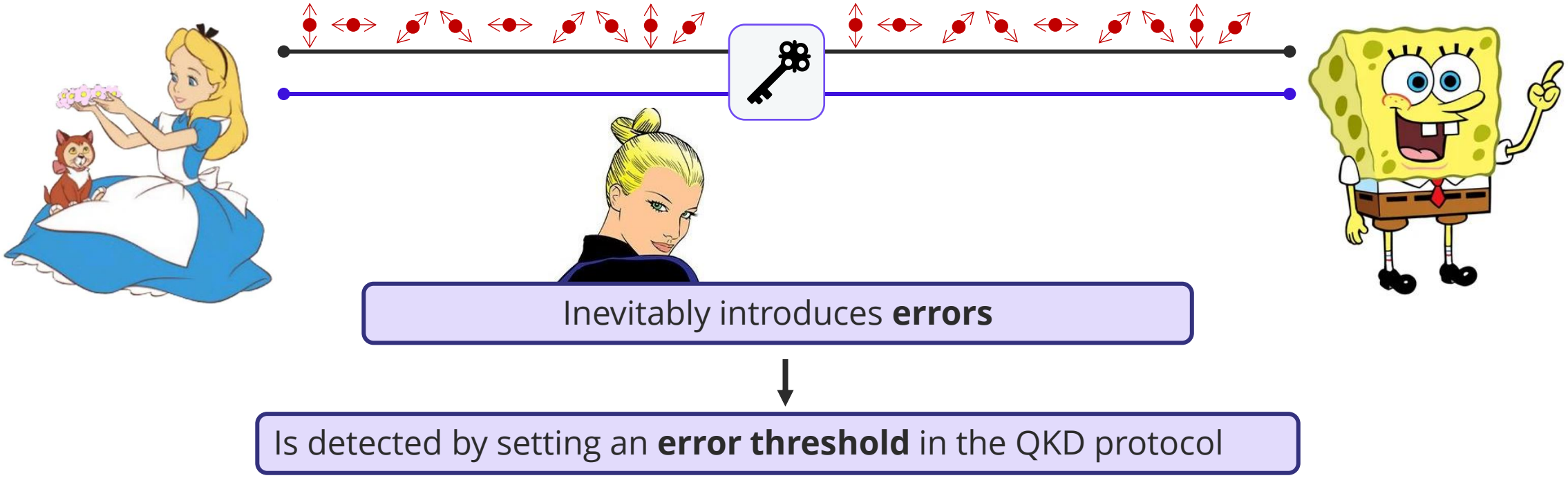


Quantum Key Distribution protocol





Eavesdropping attack and security



- Errors < threshold → **security is improved** reducing the final key length
- Errors > threshold → protocol is **aborted**
- In both cases, **no useful information** is acquired by Eve in the end.

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Thank you