TensorDay 2025
Povo 29 October 2025

Tensors meet… fluid mechanics

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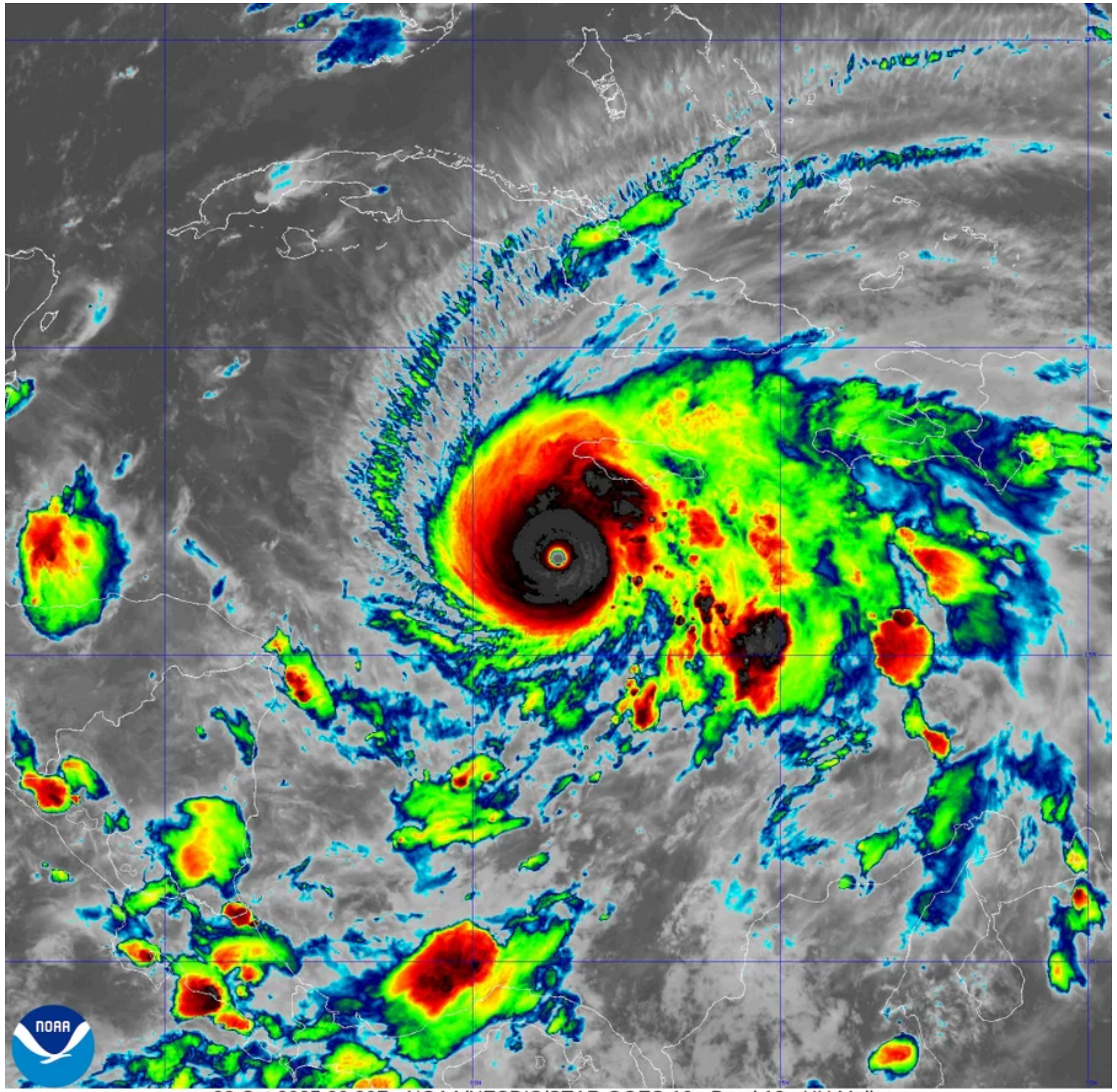
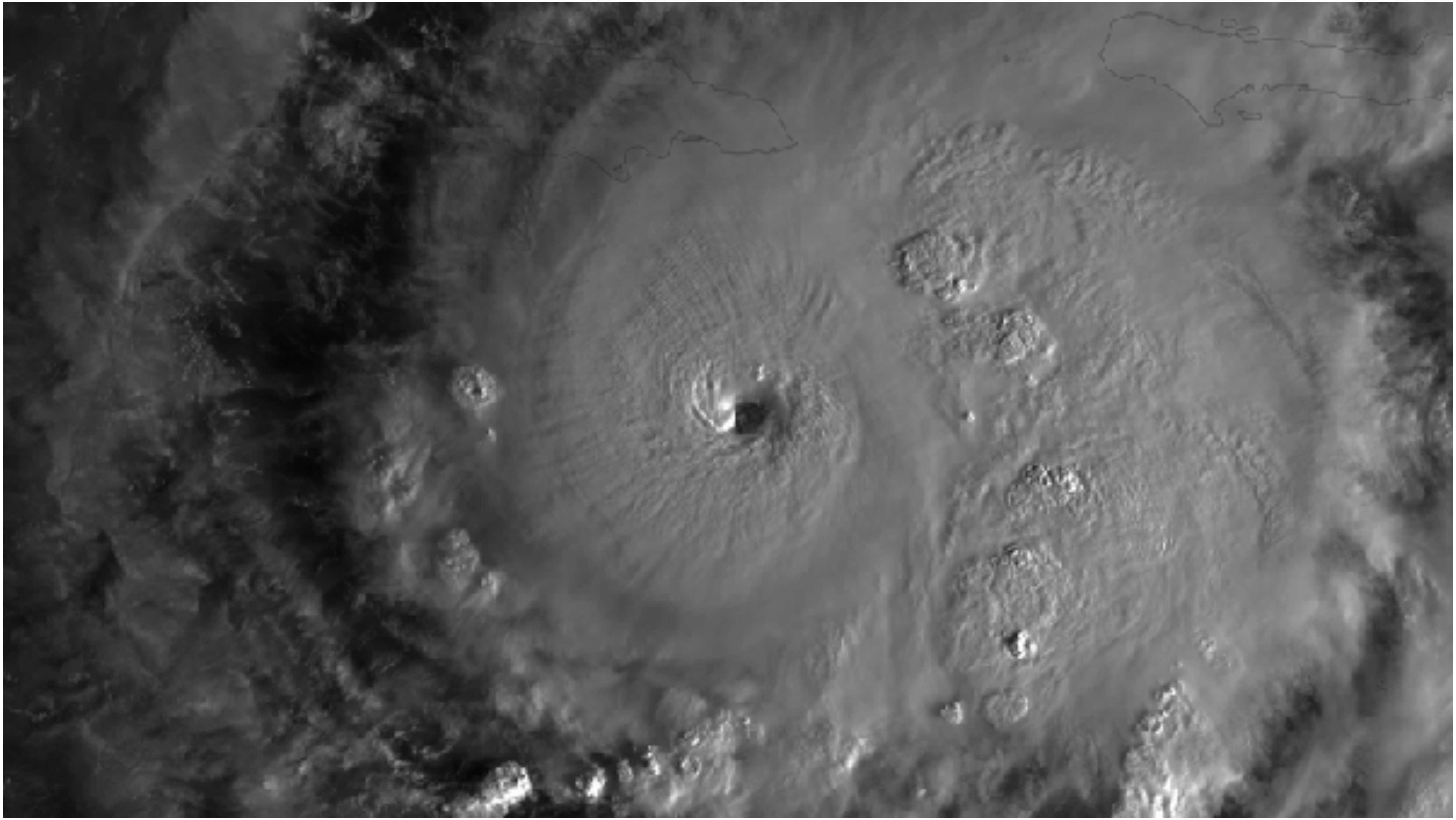


28 Oct 2025 00:20Z - NOAA/NESDIS/STAR GOES-19 - Band 13 - HU Melissa



Key Messages for Hurricane Melissa

Advisory 31: 11:00 PM EDT Tue Oct 28, 2025

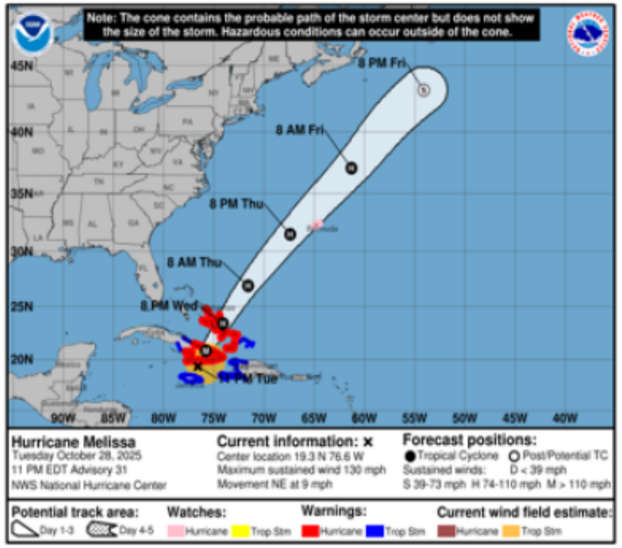
1. Jamaica: Although Melissa is pulling away from the island, deadly hazards remain including downed power lines and flooded areas. Ensure generators are properly ventilated and placed outside at least 20 feet away from doors, windows, and garages to avoid carbon monoxide poisoning. If you are cleaning up storm damage, be careful when using chainsaws and power tools, and drink plenty of water to avoid heat exhaustion.

2. Haiti and the Dominican Republic: Catastrophic flash flooding and landslides are expected across southwestern Haiti and southern portions of the Dominican Republic during the next day or so. In Haiti, extensive damage and isolation of communities is likely. Tropical storm conditions are expected into Wednesday.

3. Eastern Cuba: Life-threatening storm surge, flash flooding and landslides, and extremely damaging hurricane winds are likely through Wednesday morning. Seek safe shelter now.

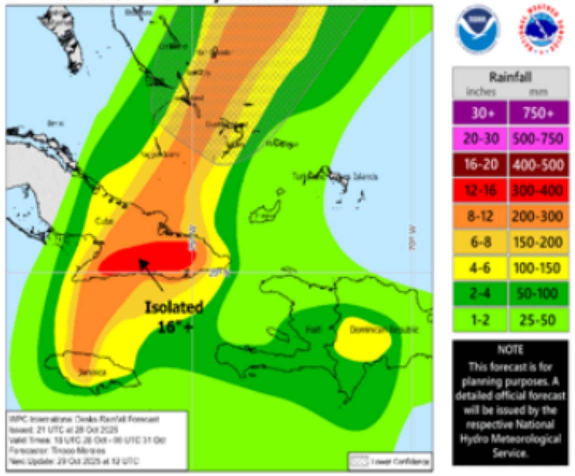
4. Southeastern and Central Bahamas and the Turks and Caicos: Hurricane conditions, life-threatening storm surge, and heavy rainfall are expected across portions of the southeastern and central Bahamas on Wednesday. Complete preparations by tonight and follow local official guidance. Tropical storm conditions, heavy rains, and significant storm surge are expected in the Turks and Caicos Islands on Wednesday.

5. Bermuda: Hurricane conditions and heavy rainfall are possible in Bermuda beginning Thursday or Thursday night, where a Hurricane Watch is in effect.



Hurricane Melissa

Additional 2-Day Rainfall Forecast



https://www.nhc.noaa.gov/

What is a fluid?

“A portion of matter that cannot withstand any tendency by applied forces to deform it in a way which leaves the volume unchanged.

A simple fluid may offer resistance to attempts to deform it, but this resistance cannot prevent the deformation from occurring, or equivalently, the resisting force vanishes with the *rate* of deformation”

(Bachelor 1967)

Examples of fluids

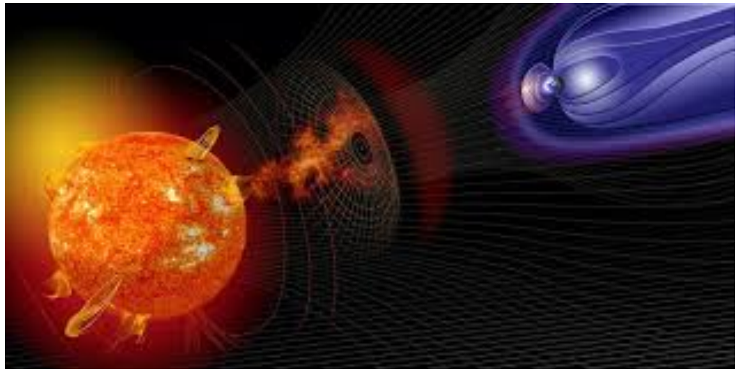










Dealing of fluids






Leonhard Euler («Eulero»)
Basilea 1707 – San Pietroburgo 1783)

INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM

IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI
Impenſis Academiae Imperialis Scientiarum
1768.

https://it.wikipedia.org/wiki/Eulero

PRINCIPES GÉNÉRAUX DU MOUVEMENT DES FLUIDES.

PAR M. EULER.

I.

Ayant établi dans mon Mémoire précedent les principes de l'équilibre des fluides le plus généralement, tant à l'égard de la diverſe qualité des fluides, que des forces qui y puiſſent agir ; je me propoſe de traiter ſur le même pied lè mouvement des fluides, & de rechercher les principes géneraux, ſur lesquels toute la ſcience du mouvement des fluides eſt fondée. On comprend aiſément que cette matiere eſt beaucoup plus difficile, & qu'elle renferme des recherches incomparablement plus profondes : cependant j'eſpère d'en venir auſſi heureuſement à bout, de ſorte que s'il y reſte des difficultés, ce ne ſera pas du côté du méchanique, mais uniquement du côté de l'analytique : cette ſcience n'étant pas encore portée à ce degré de perfection, qui ſeroit néceſſaire pour développer les formules analytiques, qui renferment les principes du mouvement des fluides.

II. Il s'agit donc de découvrir les principes, par lesquels on puiſſe déterminer le mouvement d'un fluide, en quelque état qu'il ſe trouve, & par quelques forces qu'il ſoit ſollicité. Pour cet effet examinons en détail tous les articles, qui conſtituent le ſujet de nos recherches, & qui renferment les quantités tant connues qu'inconnues. Et d'abord la nature du fluide eſt ſuppoſée connue, dont il faut conſidérer les diverſes eſpeces : le fluide eſt donc, ou incompreſſible, ou compreſſible. S'il n'eſt pas ſuſceptible de compreſſion, il faut diſtinguer deux cas, l'un où toute la maſſe eſt compoſée de parties homogenes, dont la denſité eſt partout & demeure toujours la même, l'autre

tre

Euler's equations

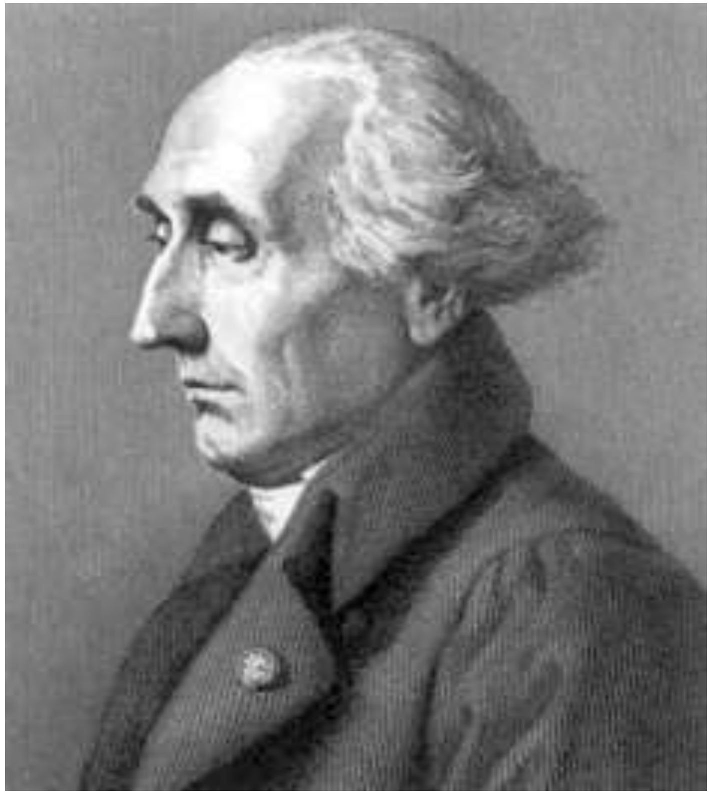
$$\left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right) = X$$

$$\left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right) = Y$$

$$\left(\frac{dw}{dt}\right) + u\left(\frac{dw}{dx}\right) + v\left(\frac{dw}{dy}\right) + w\left(\frac{dw}{dz}\right) = Z$$

$$\boldsymbol{v} = (u\,, v\,, w\,) = \left(v_1, v_2, v_3\right)$$

$$\frac{dv_i}{dt} = \left(\frac{\partial}{dt} + \boldsymbol{v}\cdot\nabla\, v_i\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}$$



Giuseppe Luigi Lagrang
(Torino 1736 – Parigi 1813

ŒUVRES

DE LAGRANGE,

PUBLIÉES PAR LES SOINS

DE M. J.-A. SERRET,

SOUS LES AUSPICES

DE SON EXCELLENCE
LE MINISTRE DE L'INSTRUCTION PUBLIQUE.

TOME QUATRIÈME.

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DE L'ÉCOLE IMPÉRIALE POLYTECHNIQUE, DU BUREAU DES LONGITUDES,
SUCCESSEUR DE MALLET-BACHELIER,
Quai des Augustins, 55.

M DCCC LXIX

MÉMOIRE

SUR LA

THÉORIE DU MOUVEMENT DES FLUIDES (*).

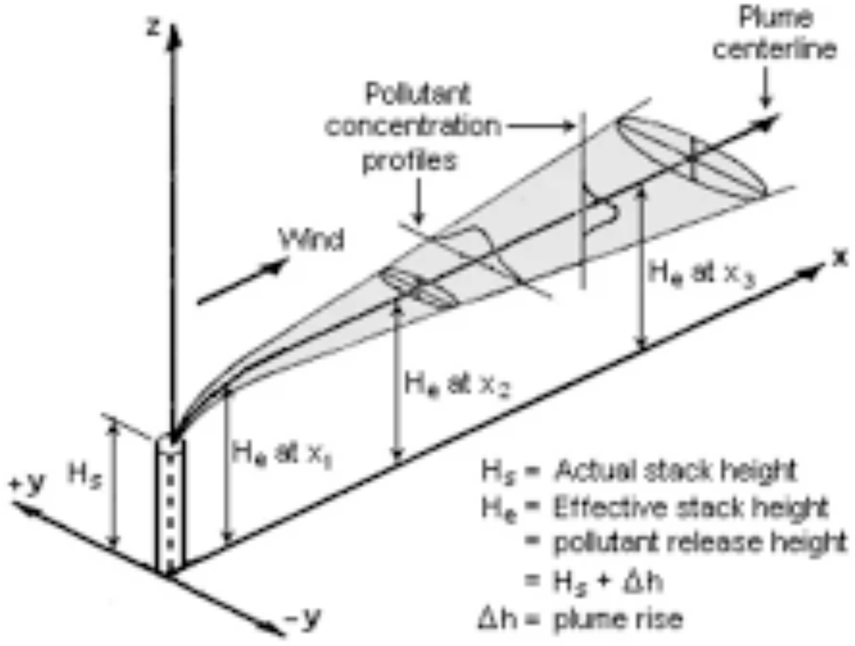
(*Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin*, année 1781.)

Depuis que M. d'Alembert a réduit à des équations analytiques les vraies lois du mouvement des fluides, cette matière est devenue l'objet d'un grand nombre de recherches qui se trouvent répandues dans les *Opuscules* de M. d'Alembert, et dans les Recueils de cette Académie et de celle de Pétersbourg. La Théorie générale a été beaucoup perfectionnée dans ces différentes recherches; mais il n'en est pas de même de la partie de cette Théorie qui concerne la manière de l'appliquer aux questions particulières. M. d'Alembert paraît même porté à croire que cette application est impossible dans la plupart des cas, surtout lorsqu'il s'agit du mouvement des fluides qui coulent dans des vases.

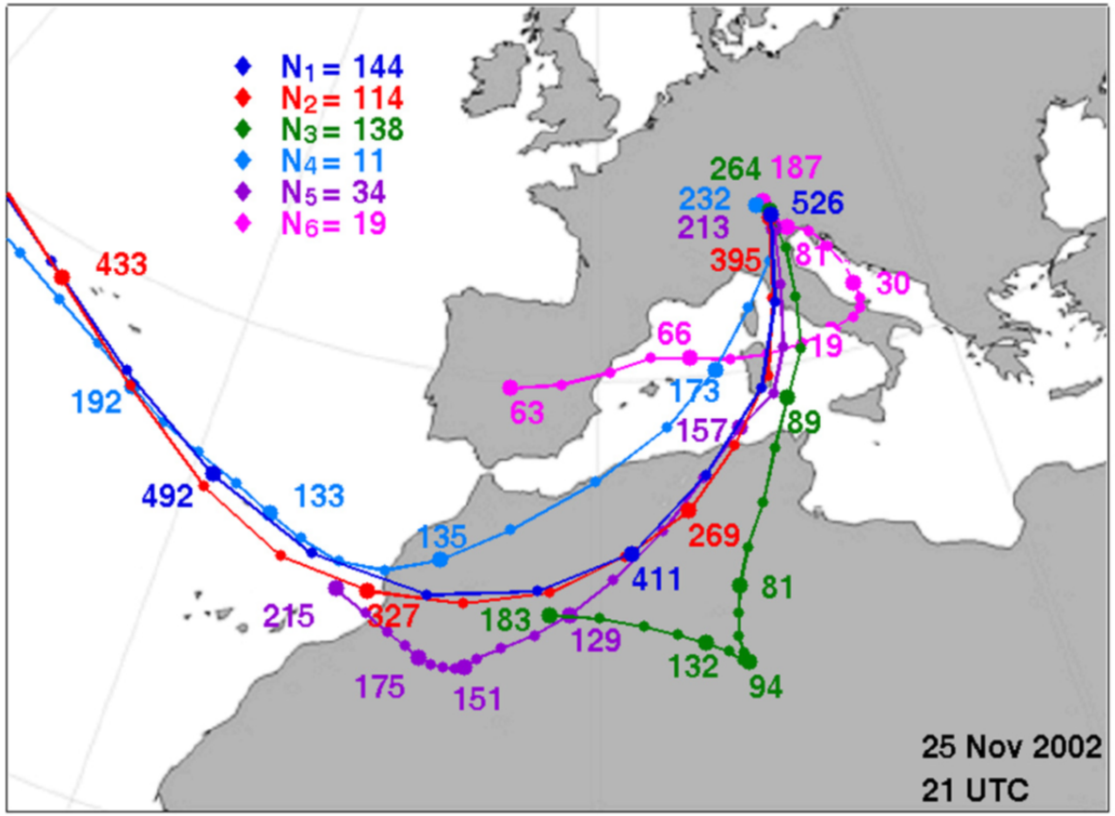
Après avoir soigneusement étudié tout ce qui a déjà été écrit sur la Théorie rigoureuse du mouvement des fluides, je me suis appliqué à lever, ou du moins à diminuer les difficultés qui ont retardé les progrès de cette Théorie, et ont obligé les Géomètres à se contenter, pour la solution des Problèmes les plus simples, de méthodes indirectes, ou fon-

(*) Lu le 22 novembre 1781.

Eulerian approach



Lagrangian approach



Rotation of the coordinate system

Now suppose that the coordinate system is rigidly rotated to a new position $O\bar{1}\bar{2}\bar{3}$ as shown in Fig. 2.2 and the new coordinates of P are $\bar{x}_1, \bar{x}_2, \bar{x}_3$. The rotation can be specified by giving the angles between the old and new axes. Let l_{ij} be the cosine of the angle between the old O axis Oi and the new one $O\bar{j}$, then the new coordinates are related to the old by the formulae

$$\bar{x}_j = l_{1j}x_1 + l_{2j}x_2 + l_{3j}x_3, \qquad j = 1, 2, 3,$$

and conversely

$$x_i = l_{i1}\bar{x}_1 + l_{i2}\bar{x}_2 + l_{i3}\bar{x}_3, \qquad i = 1, 2, 3.$$

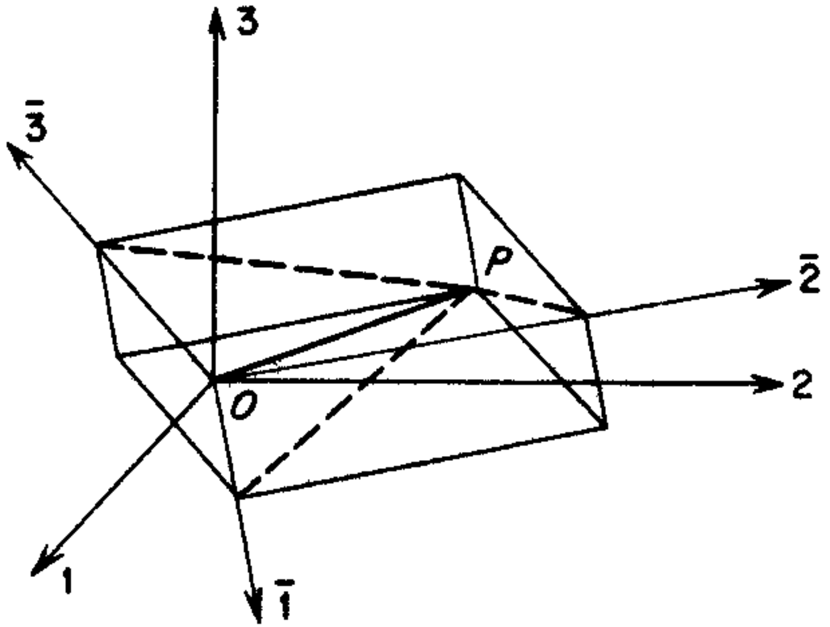


Fig. 2.2

(Aris 1962)

2.41. Second order tensors

The vector or first order tensor was defined as an entity with three components which transformed in a certain fashion under rotation of the coordinate frame. We define a *second order Cartesian tensor* similarly as an entity having nine components A_{ij}, $i, j = 1, 2, 3$, in the Cartesian frame of reference $O123$ which on rotation of the frame of reference to $O\bar{1}\bar{2}\bar{3}$ become

$$\bar{A}_{pq} = l_{ip} l_{jq} A_{ij}. \tag{2.41.1}$$

By the orthogonality properties of the direction cosines l_{rs} we have the inverse transformation

$$A_{ij} = l_{ip} l_{jq} \bar{A}_{pq}. \tag{2.41.2}$$

To establish that a given entity is a second order tensor we have to demonstrate that its components transform according to Eq. (2.41.1). A valuable means of establishing tensor character is the quotient rule which will be discussed later in Section 2.6.

If **a** and **b** are two vectors the set of nine products $a_i b_j = A_{ij}$ is a second order tensor, for

$$\begin{aligned} \bar{A}_{pq} &= \bar{a}_p \bar{b}_q = l_{ip} a_i l_{jq} b_j = l_{ip} l_{jq} (a_i b_j) \\ &= l_{ip} l_{jq} A_{ij}. \end{aligned} \tag{2.42.2}$$

An important example of this is the momentum flux tensor for a fluid. If ρ is the density and **v** the velocity, ρv_i is the i^{th} component in the direction Oi. The rate at which this momentum crosses a unit area normal to Oj is $\rho v_i v_j$.

2.44. Contraction and multiplication

The operation of identifying two indices of a tensor and so summing on them is known as contraction.

The suffix notation makes quite clear just which contraction is involved. However, the notation of a scalar product is sometimes useful. In this notation the tensor with components $A_{ij}B_{jk}$ is written $\mathbf{A}\cdot\mathbf{B}$ the summation being over adjacent suffixes.

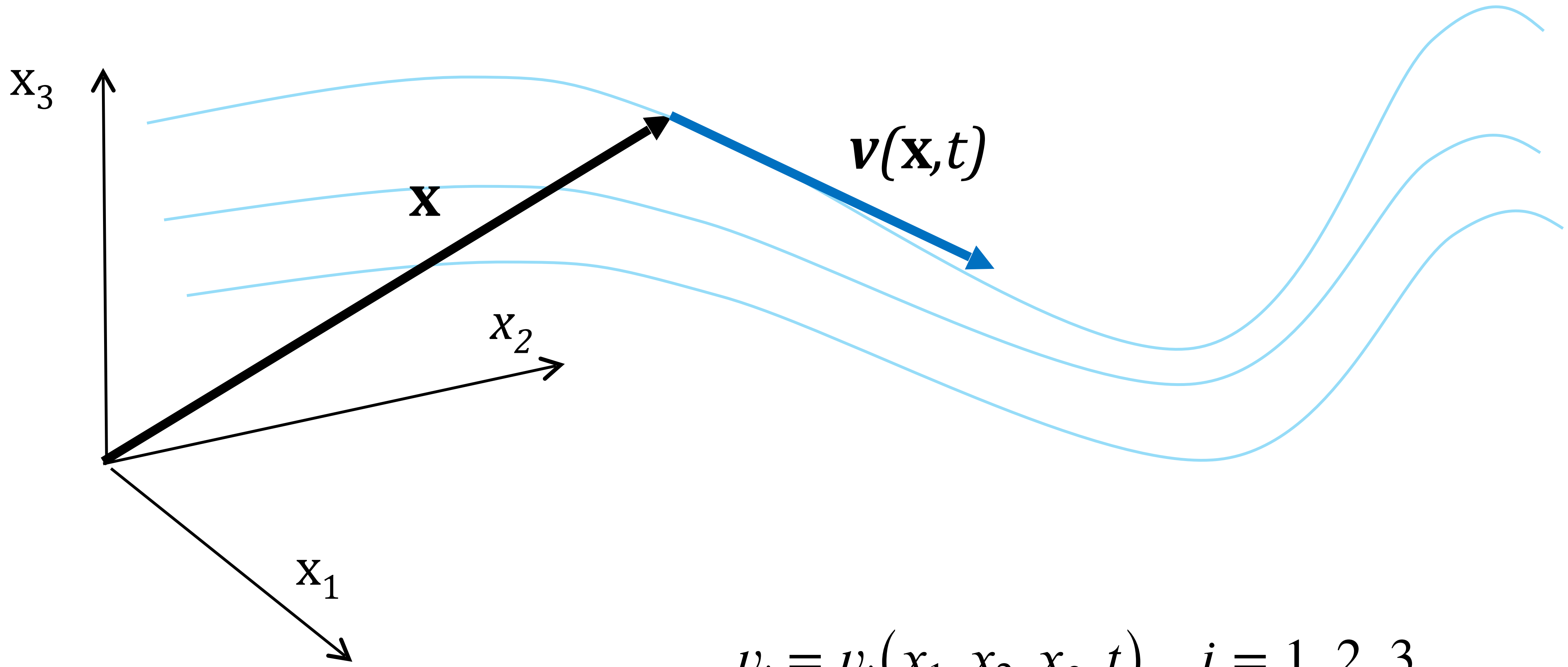
The product $A_{ij}a_j$ of a vector $\mathbf{a}$ and tensor $\mathbf{A}$ is a vector whose i^{th} component is $A_{ij}a_j$. Another possible product of these two is $A_{ij}a_i$. These may be written $\mathbf{A}\cdot\mathbf{a}$ and $\mathbf{a}\cdot\mathbf{A}$ respectively.

The doubly contracted product $A_{ij}B_{ji}$ is a scalar, and this may be written $\mathbf{A}:\mathbf{B}$.

2.62. The quotient rule

We have constantly remarked that to prove that a given set of quantities forms the set of the components of a tensor requires that we show that they transform according to the rule of tensor transformation. A short cut in establishing tensorial character is the so-called quotient rule. The simple case we shall prove is as follows: If A_{ij} $i, j = 1, 2, 3$ are nine quantities and $\mathbf{b}$ and $\mathbf{c}$ are vectors, $\mathbf{b}$ being quite independent of the A_{ij}, and $A_{ij}b_j = c_i$, then the A_{ij} are components of a tensor $\mathbf{A}$. The value of this is that a relation $\mathbf{A} \cdot \mathbf{b} = \mathbf{c}$ may arise in the study of a physical situation in which it is known that $\mathbf{b}$ and $\mathbf{c}$ are vectors. Then the quotient rule establishes that $\mathbf{A}$ is a tensor and we are now assured that the equation holds in all coordinate frames.

The velocity vector field



$$v_i = v_i\left(x_1, x_2, x_3, t\right) \quad i = 1,\, 2,\, 3$$

The velocity gradient tensor

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

$$= e_{ij} + \Omega_{ij}$$

$\boldsymbol{e}$ deformation or rate of strain tensor

Ω rate of rotation tensor

The rate of strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

$$Tr(\boldsymbol{e}) = \frac{\partial v_i}{\partial x_i} = \nabla \bullet \boldsymbol{v} = \Theta$$ First invariant

The rate of rotation tensor

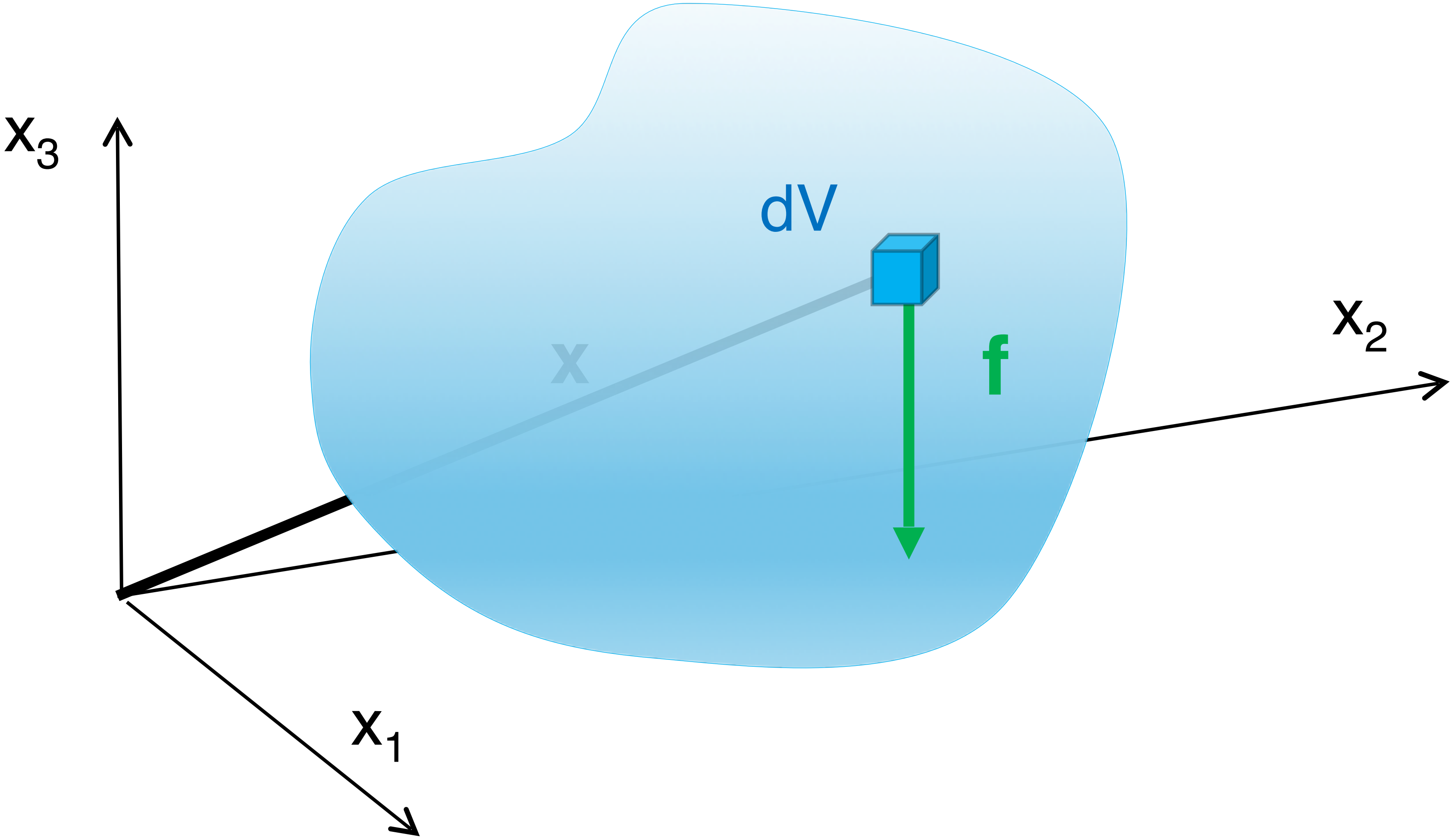
$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) & 0 \end{pmatrix}$$

Vorticity

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\boldsymbol{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\boldsymbol{e}_2 + \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\boldsymbol{e}_3$$

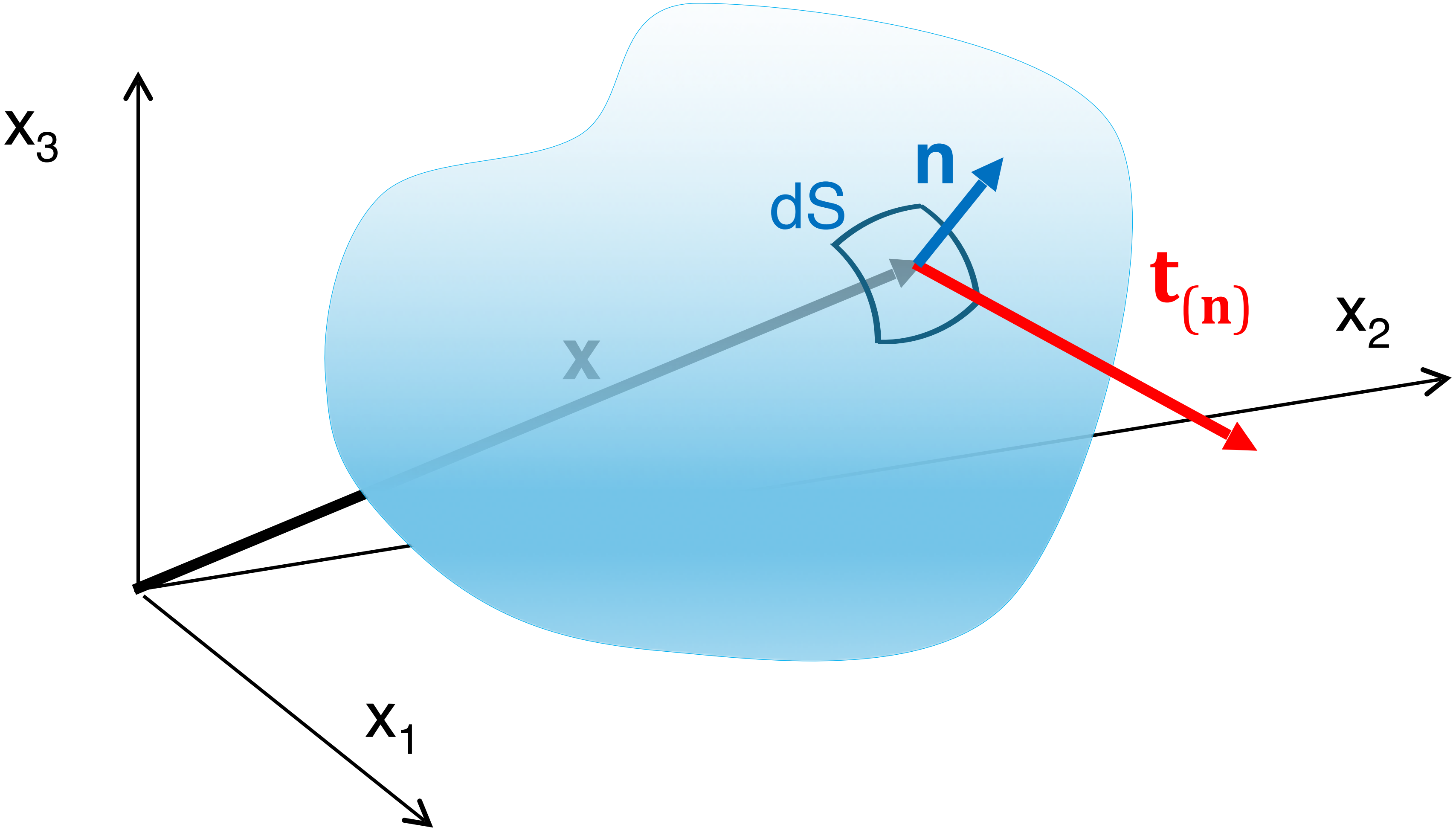
$$\Omega_{ij} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_1 & \omega_2 \\ \omega_1 & 0 & -\omega_3 \\ -\omega_2 & \omega_3 & 0 \end{pmatrix}$$

External (or *body* or *long-range*) forces



Total external force exerted on the volume V : $\int_V \rho \boldsymbol{f} \, dV$

Internal (or *surface* or short-range) forces



Total force exerted by external forces on the volume V through its bounding surface S: $\int t_{(n)} dS$

Cauchy's stress principle

Augustin-Louis
Cauchy (1789 – 1857)

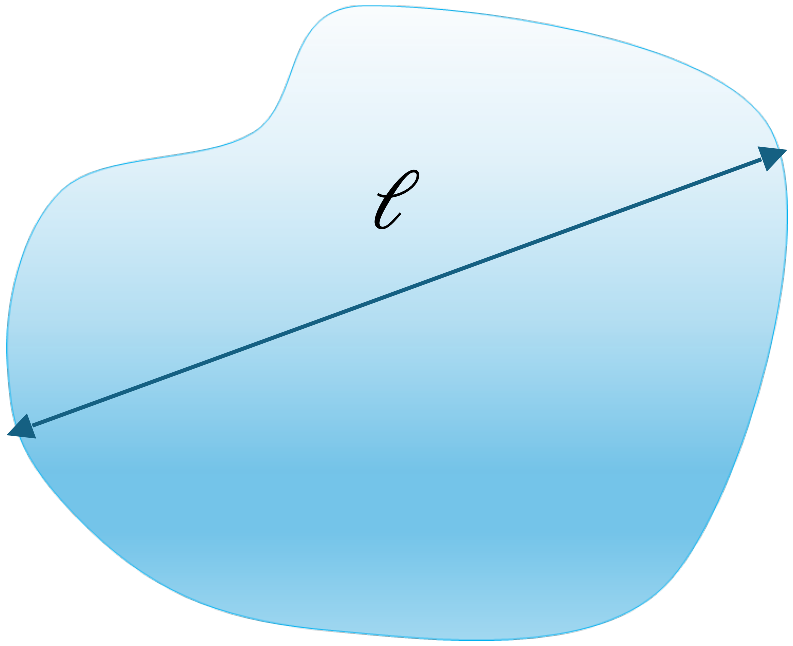
The stress $\boldsymbol{t}_{(n)}$ is a function of the position $\boldsymbol{x}$, the time t, and the orientation $\boldsymbol{n}$ of the surface element:

$$\boldsymbol{t}_{(n)}(\boldsymbol{x},\boldsymbol{n},t)$$

Conservation of linear momentum

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \int_V \rho \boldsymbol{f}\, dV + \int_S \boldsymbol{t}_{(\boldsymbol{n})} dS$$

-> Similar reasoning for angular momentum



Let ℓ be a characteristic linear dimension of a fluid body having a volume V bounded by a surface S.

Then the volume will be ~ℓ^3, while the bounding surface will be ~ℓ^2, with the proportionality constants depending only on the shape.

Let the body shrink to a point, while preserving its shape: then the volume integrals will decrease as ℓ^3 , while the surface integral will decrease as ℓ^2 .

From conservation of linear momentum it follows that:

$$\lim_{\ell \to 0} \frac{1}{\ell^2} \int_S \boldsymbol{t}_{(\boldsymbol{n})} dS = 0$$

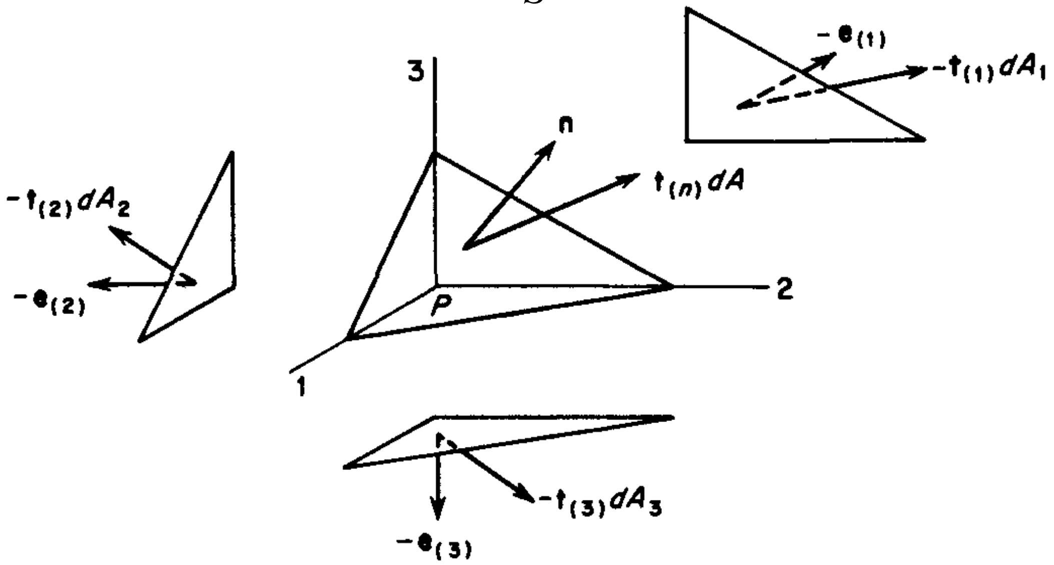
In other words, <u>the stress are always locally in equilibrium.</u>

The stress tensor

To elucidate the nature of the stress system at a point P we consider a small tetrahedron with three of its faces parallel to the coordinate planes through P and the fourth with normal $\mathbf{n}$ (see Fig. 5.1). If dA is the area of the slant face, the areas of the faces perpendicular to the coordinate axis Pi is $dA_i = n_i\, dA$. The outward normals to these faces are $-\mathbf{e}_{(i)}$ and we may denote the stress vector over these faces by $-\mathbf{t}_{(i)}$. ($\mathbf{t}_{(i)}$ denotes the stress vector when $+\mathbf{e}_{(i)}$ is the outward normal.) Then applying the principle of local equilibrium to the stress forces when the tetrahedron is very small we have

$$\int_S t_{(n)} dS = \mathbf{t}_{(n)}\, dA - \mathbf{t}_{(1)}\, dA_1 - \mathbf{t}_{(2)}\, dA_2 - \mathbf{t}_{(3)}\, dA_3$$

$$= (\mathbf{t}_{(n)} - \mathbf{t}_{(1)} n_1 - \mathbf{t}_{(2)} n_2 - \mathbf{t}_{(3)} n_3)\, dA = 0.$$



Now let T_{ji} denote the i^{th} component of $\mathbf{t}_{(j)}$ and $t_{(n)i}$ the i^{th} component of $\mathbf{t}_{(n)}$ so that this equation can be written

$$t_{(n)i} = T_{ji} n_j. \tag{5.12.2}$$

However, $\mathbf{t}_{(n)}$ is a vector and $\mathbf{n}$ is a unit vector quite independent of the T_{ji} so that by the quotient rule the T_{ji} are components of a second order tensor $\mathbf{T}$. In dyadic notation we might write

$$\mathbf{t}_{(n)} = \mathbf{n} \cdot \mathbf{T}. \tag{5.12.3}$$

This tells us that the system of stresses in a fluid is not so complicated as to demand a whole table of the functions $\mathbf{t}_{(n)}(\mathbf{x}, \mathbf{n})$ at any given instant, but that it depends rather simply on $\mathbf{n}$ through the nine quantities $T_{ji}(\mathbf{x})$. Moreover, because these are components of a tensor, any equation we derive with them will be true under any rotation of the coordinate axes.

Inserting Eq. (5.12.2) in Eq. (5.11.3) and using Green's theorem we have

$$\frac{d}{dt}\iiint_V \rho v_i\, dV = \iiint_V \rho \frac{dv_i}{dt}\, dV = \iiint_V \rho f_i\, dV + \iint_S T_{ji} n_j\, dS$$

$$= \iiint_V [\rho f_i + T_{ji,j}]\, dV.$$

However, since V is an arbitrary volume this equation is only satisfied if

$$\rho \frac{dv_i}{dt} = \rho f_i + T_{ji,j} \tag{5.12.4}$$

or

$$\boxed{\rho \mathbf{a} = \rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T},} \tag{5.12.5}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the acceleration. This is Cauchy's equation of motion. It holds for any continuum no matter how the stress tensor $\mathbf{T}$ is connected with the rate of strain.

5.14. Hydrostatic pressure

If the stress system is such that an element of area always experiences a stress normal to itself and this stress is independent of the orientation, the stress is called *hydrostatic*. All fluids at rest exhibit this stress behavior. It implies that $\mathbf{n} \cdot \mathbf{T}$ is always proportional to $\mathbf{n}$ and that the constant of proportionality is independent of $\mathbf{n}$. Let us write this constant $-p$, then

$$n_i T_{ij} = -p n_j. \tag{5.14.1}$$

However, this equation means that any vector is a characteristic vector of $\mathbf{T}$ which must therefore be spherical. Thus

$$T_{ij} = -p\delta_{ij} \tag{5.14.2}$$

for a state of hydrostatic stress.

For a compressible fluid at rest, p may be identified with the pressure of classical thermodynamics. On the assumption that there is local thermodynamic equilibrium even when the fluid is in motion this concept of stress may be retained. For an incompressible fluid the thermodynamic, or more correctly thermostatic, pressure cannot be defined except as the limit of pressure in a sequence of compressible fluids. We shall see later that it has to be taken as an independent dynamical variable.

The stress tensor may always be written

$$T_{ij} = -p\delta_{ij} + P_{ij},$$

and P_{ij} is called the viscous stress tensor. The mean of the three stresses T_{11}, T_{22}, and T_{33} is

$$\tfrac{1}{3} T_{ii} = -p + \tfrac{1}{3} P_{ii}. \tag{5.14.2}$$

In the case of hydrostatic stress, where P_{ij} vanishes, this mean stress equals the thermostatic pressure. We shall show later that for the incompressible Newtonian fluid this is also true, but a distinction must be made in general between the mean stress and the pressure. A perfect fluid is one for which P_{ij} vanishes identically.

5.21. The Stokesian fluid

George Gabriel Stokes
(1819 –1903)

I. The stress tensor T_{ij} is a continuous function of the deformation tensor e_{ij} and the local thermodynamic state, but independent of other kinematical quantities.
II. The fluid is homogeneous, that is, T_{ij} does not depend explicitly on $\mathbf{x}$.
III. The fluid is isotropic, that is, there is no preferred direction.
IV. When there is no deformation ($e_{ij} = 0$) the stress is hydrostatic, ($T_{ij} = -p\delta_{ij}$).

5.22. Constitutive equations of the Stokesian fluid

$$T_{ij} = (-p + \alpha)\,\delta_{ij} + \beta e_{ij} + \gamma e_{ik} e_{kj}. \tag{5.22.6}$$

p depends only on the thermodynamic state but α, β, and γ depend as well on the invariants of the rate of strain tensor. This gives ample scope for the fitting of exceedingly complex relations, but the tensorial character is prescribed by the assumptions.

If the fluid is compressible, the thermodynamic pressure is a well-defined quantity and we should take p equal to this. Then, by the fourth assumption, $\alpha = 0$ when $e_{ij} = 0$. If the fluid is incompressible, the thermodynamic pressure is not defined and pressure has to be taken as one of the fundamental dynamical variables. We are at liberty to do this in the simplest possible way so that without losing any generality we can absorb α into the pressure p and write

$$T_{ij} = -p\delta_{ij} + \beta e_{ij} + \gamma e_{ik} e_{kj}, \tag{5.22.7}$$

which insures that T reduces to the hydrostatic form when the deformation vanishes.

5.23. The Newtonian fluid

The Newtonian fluid is a linear Stokesian fluid, that is, the stress components depend linearly on the rates of deformation.

$$T_{ij} = (-p + \lambda\Theta)\delta_{ij} + 2\mu e_{ij}. \qquad (5.23.4)$$

Consider the shear flow given by

$$v_1 = f(x_2), \qquad v_2 = v_3 = 0. \qquad (5.24.1)$$

For this we have all the e_{ij} zero except

$$e_{12} = e_{21} = \tfrac{1}{2}f'(x_2). \qquad (5.24.2)$$

Thus

$$P_{12} = P_{21} = \mu f'(x_2) \qquad (5.24.3)$$

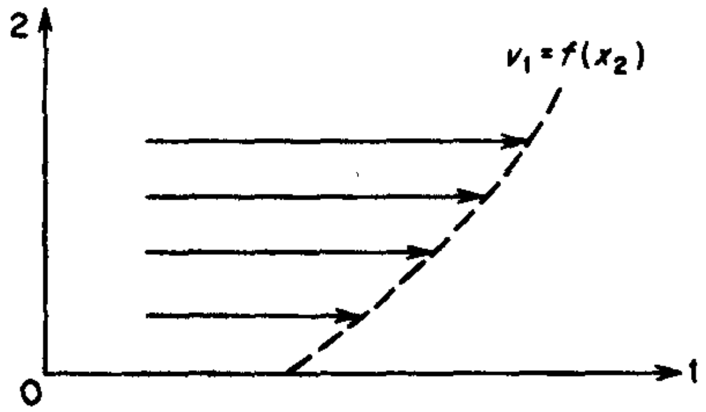


Fig. 5.2

and all the other viscous stresses are zero. This is shown in Fig. 5.2 and it is evident that μ is the proportionality constant relating the shear stress to the velocity gradient. This is the common definition of the viscosity, or more precisely the coefficient of shear viscosity, of a fluid.

6.11. Equations of motion of a Newtonian fluid

Cauchy's equation of motion is

$$\rho a_i = \rho \frac{dv_i}{dt} = \rho f_i + T_{ij,j} \tag{6.11.1}$$

for the symmetric stress tensor, which is related to the rate of strain tensor by

$$T_{ij} = (-p + \lambda\Theta)\delta_{ij} + 2\mu e_{ij}. \tag{6.11.2}$$

$$\rho \frac{dv_i}{dt} = \rho f_i - \frac{\partial p}{\partial x_i} + (\lambda + \mu)\frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{v}) + \mu \nabla^2 v_i. \tag{6.11.4}$$

Equation (6.11.4) is the i^{th} component of

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + (\lambda' + \nu)\nabla(\nabla \cdot \mathbf{v}) + \nu \nabla^2 \mathbf{v} \tag{6.11.5}$$

where $\nu = \mu/\rho$, $\lambda' = \lambda/\rho$. ν is known as the kinematic viscosity and if Stokes' relation is assumed $\lambda' + \nu = \nu/3$. For an incompressible fluid we have

Navier-Stokes equation

$$\boxed{\frac{d\mathbf{v}}{dt} = \mathbf{f} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v},} \tag{6.11.6}$$

and for an incompressible inviscid or perfect fluid the equations drop out by setting $\nu = 0$.

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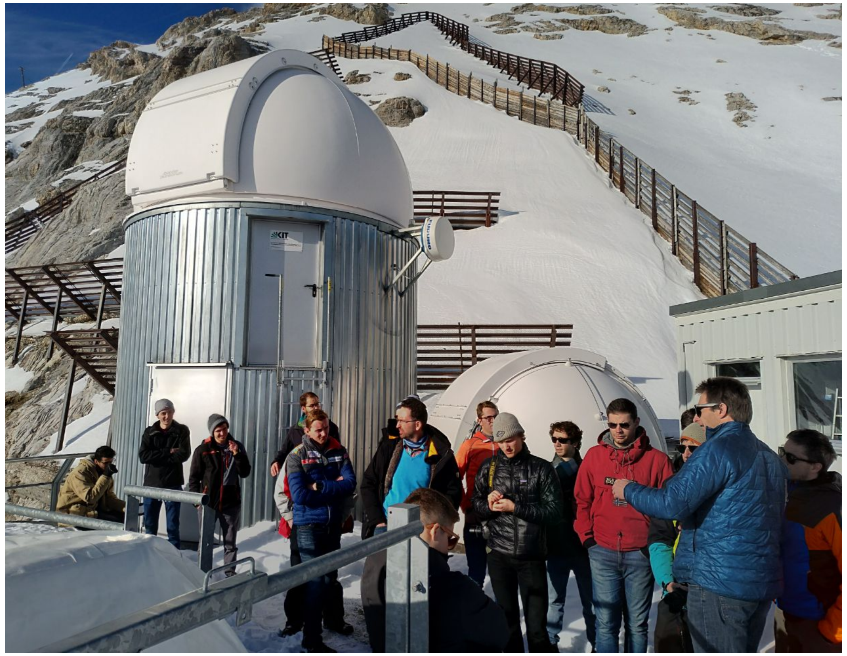

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Suggested readings

Aris, R., 1962: *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Books on Mathematics
Pdf file available at this link.

Bachelor, G. 1967: *An introduction to fluid dynamics*, Cambridge University Press.

Tennekes, H. and Lumley, J., 1972: *A First Course in Turbulence*. MIT Press

Pope, S. B. 2000: *Turbulent Flows*, Cambridge University Press.

Other resources

Galley of Fluid Motions
American Physical Society (APS), Division of Fluid Dynamics
https://gfm.aps.org/

5.15. Principal axes of stress and the notion of isotropy

The diagonal terms T_{11}, T_{22}, T_{33} of the stress tensor are sometimes called the *direct stresses* and the terms T_{12}, T_{21}, T_{31}, T_{13}, T_{23}, T_{32} the *shear stresses*. When there are no external or stress couples, the stress tensor is symmetric and we can invoke the known properties of symmetric tensors. In particular, there are three principal directions and referred to coordinates parallel to these, the shear stresses vanish. The remaining direct stresses are called the *principal stresses* and the axes *the principal axes of stress*. The mean pressure $-p$ is one third of the trace of the stress tensor and so is the mean of the principal stresses.

An *isotropic* fluid is such that a simple direct stress acting in it does not produce a shearing deformation. This is an entirely reasonable view to take for isotropy means that there is no internal sense of direction within the fluid so that a direct stress, say

$$T_{11} \neq 0, \qquad T_{ij} = 0 \qquad i, j \neq 1,$$

should not produce any differential motion in planes parallel to its line of action, in this case the axis $O1$. Another way of expressing the absence of any internally preferred direction is to say that the functional relation between stress and deformation must be independent of the orientation of the coordinate system.