TensorDay 2023
Inaugural day of the TensorDec Laboratory

21 November 2023

TENSORDEC

THE LABORATORY ON ALGEBRA, GEOMETRY AND APPLICATIONS OF TENSOR DECOMPOSITIONS
OF THE UNIVERSITY OF TRENTO

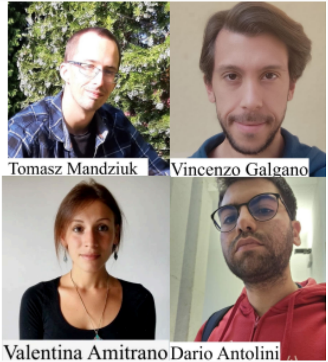
Since 2021, the TensorDec Laboratory is a recipient for activities of research, teaching, mentoring and networking about algebra, geometry and applications of tensor decompositions

tensordec.maths.unitn.it


Lost in the labyrinth of Tensors.
I'm clueless!
Let's knock at this door and hope
a thesis is a key out
Prof. X


Edoardo Ballico
Elisa Postinghel
Alessandra Bernardi
Alessandro Oneto
Mima Stanojkovski

TensorDec Laboratory: who we are



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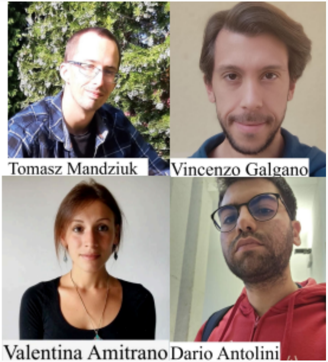
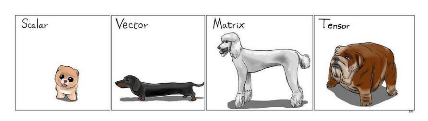


Why should you knock at those doors? What are tensors and what do we do with them?

What are tensors?

Tensors are multidimensional boxes for organizing numbers, much like matrices are boxes for organizing numbers in two ways.



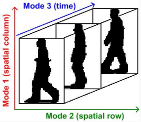
Like matrices, tensors are extremely useful and versatile.

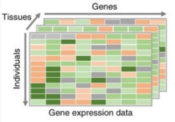
They can be viewed as:

- Tables → Boxes of numbers with a certain order.
- Linear → Multilinear maps.
- Elements of the Matrix → Tensor space.

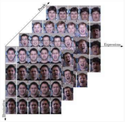
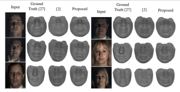
Tensors: as boxes of numbers

Multidimensional numerical arrays:

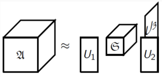
Used to store tons of data 



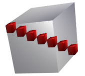
extract meaningful information


Algorithms for Tensor Decompositions (come to our courses/masterclasses):

- Tucker Decomposition: $\mathcal{U} = (U_1, U_2, U_3)\mathcal{C}$: 

- Rank Decomposition (parafac/candecomp/cpd):

$\mathcal{U} = \sum_{i=1}^{r} u_{i,1} \otimes u_{i,2} \otimes u_{i,3}$: 

Tensors: as multilinear maps

A matrix $M \in W \otimes V$ is equivalent to a linear map $f_M : V \to W$.

Multilinear maps

$$
\begin{array}{cccc}
MaMu_2 : & (\mathbb{C}^{2,2})^{\times 2} & \to & \mathbb{C}^{2,2} \\
& \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) & \mapsto & \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}
\end{array}
$$

$$
\begin{aligned}
(\mathbb{C}^{2,2})^{\otimes 3} \ni MaMu_2 = {} & (a_{11} \otimes b_{11} + a_{12} \otimes b_{21}) \otimes c_{11} + (a_{11} \otimes b_{12} + a_{12} \otimes b_{22}) \otimes c_{12} \\
& + (a_{21} \otimes b_{11} + a_{22} \otimes b_{21}) \otimes c_{21} + (a_{21} \otimes b_{12} + a_{22} \otimes b_{22}) \otimes c_{22}
\end{aligned}
$$

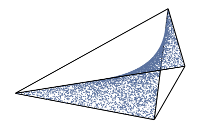
A decomposition of $MaMu_2$ highlights an algorithm.

The rank decomposition highlights the best one. Rank = Complexity → Complexity Theory.

If the algorithm is encoded in a Quantum Circuit: Complexity = Entanglement of the state → Quantum Information.

Tensors: as points in the space

Points of a tensor space: matrix $M \in W \otimes V$, tensor $\mathcal{U} \in V_1 \otimes \cdots \otimes V_d$



This perspective allows to study tensors by their common properties, using global techniques like algebraic-geometric methods (both symbolic and numerical) to extract shared information among them.

- Mima Stanojkovski: "Tensors in finite group theory".
- Alessandro Oneto: "Tensors into algebraic statistical models".
- Elisa Postinghel: "Tensors and polynomial interpolation".
- Alessandra Bernardi: "Symbolic algorithms for Tensor Decomposition".
- Edoardo Ballico: "Algebraic Geometry aspects of Tensors".

TensorDec Laboratory: studying with us - Mentoring

We mentor theses

Algebra


Mima Stanojkovski


Alessandra Bernardi

- Commutative and non commutative algebra, Group theory, Scheme theory.
- *Interdisciplinary and Industrial*: Quantum physics, Data science.

Algebraic Geometry


Alessandro Oneto


Elisa Postinghel


Edoardo Ballico

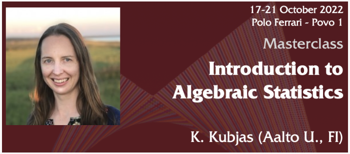
- Birational geometry, Computational geometry, Teaching.
- *Interdisciplinary and Industrial*: Data science, Algebraic statistics.

TensorDec Laboratory: what we do - Teaching

Courses (Mathematics for Data Science, Q@TN):

- *Tensor Decompositions for Big Data Analysis (A. Bernardi)*
- *Geometry and Topology for Data Analysis (A. Oneto)*

We organize annual **Masterclasses** taught by international guests:



TensorDec Laboratory: working with us - Networking

PhD opportunities (more than the standard call):

- Industrial PhD (just closed)
- Transdisciplinary Doctoral Program Q@TN
- Horizon: Marie Curie Double Degree (IT & abroad):

TENORS
Tensor modEliNg, geOmetRy and optimiSation
Marie Skłodowska-Curie Doctoral Network
2024-2027

Tensors are nowadays ubiquitous in many domains of applied mathematics, computer science, signal processing, data processing, machine learning and in the emerging area of quantum computing. TENORS aims at fostering cutting-edge research in tensor sciences, stimulating interdisciplinary and intersectoriality knowledge developments between algebraists, geometers, computer scientists, numerical analysts, data analysts, physicists, quantum scientists, and industrial actors facing real-life tensor-based problems.

Partners:

1. Inria, Sophia Antipolis, France (B. Mourrain, A. Mantzaflaris)
2. CNRS, LAAS, Toulouse, France (D. Henrion, V. Magron, M. Skomra)
3. NWO-I/CWI, Amsterdam, the Netherlands (M. Laurent)
4. Univ. Konstanz, Germany (M. Schweighofer, S. Kuhlmann, M. Michałek)
5. MPI, Leipzig, Germany (B. Sturmfels, S. Telen)
6. Univ. Tromsoe, Norway (C. Riener, C. Bordin, H. Munthe-Kaas)
7. Univ. degli Studi di Firenze, Italy (G. Ottaviani)
8. Univ. degli Studi di Trento, Italy (A. Bernardi, A. Oneto, I. Carusotto)
9. CTU, Prague, Czech Republic (J. Marecek)
10. ICFO, Barcelona, Spain (A. Acin)
11. Artelys SA, Paris, France (M. Gabay)

Associate partners:

1. Quandela, France
2. Cambridge Quantum Computing, UK.
3. Bluetensor, Italy.
4. Arva AS, Norway.
5. HSBC Lab., London, UK.

15 PhD positions (2024-2027)

(recruitment expected around Oct. 2024)

Scientific coord: B. Mourrain
Adm. manager: Linh Nguyen 1

TensorDec Laboratory: working with us - Networking

Post Doc opportunities: We are members of the
Italian Network for Applied and Birational Algebraic Geometry (INABAG)

https://sites.google.com/unitn.it/inabag/

Tensors and polynomial interpolation

Let $R_d := \mathbb{R}[x_1, x_2]_d$ be the space of polynomials of degree $\leq d$.
Let $S = \{p_1, \ldots, p_s\}$ be a set of distinct points in $\mathbb{R}^2$.

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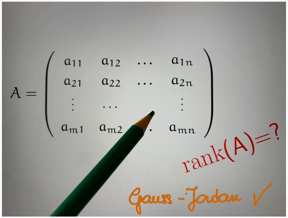
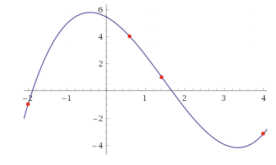


Polynomial interpolation problems:

- **Simple points**:
  $\mathcal{L}_d(S) = \{f \in R : f(p_i) = 0, \forall i = 1, \ldots, s\} \subseteq R$



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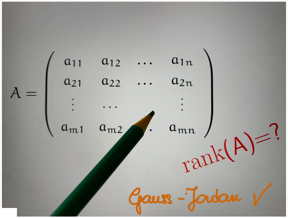

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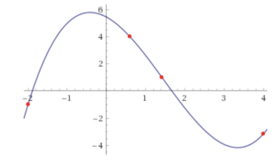

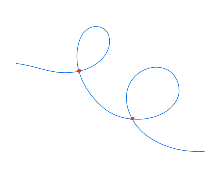

- **Double points**:
  $\mathcal{L}_d(2S) = \{F \in R_d : f(p_i) = \frac{\partial}{\partial x_1} f(p_j) = \frac{\partial}{\partial x_2} f(p_j) = 0, \forall j = 1, \dots, s\} \subseteq R$

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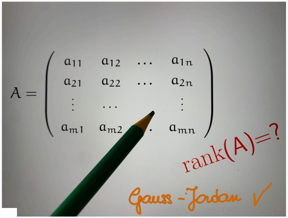

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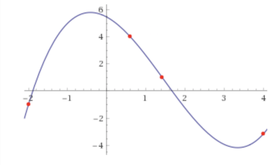

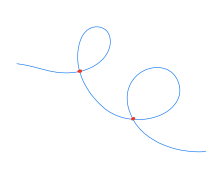

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- **Multiple points**:
  $\mathcal{L}_d(mS) = \ldots$

Tensors and polynomial interpolation

$R_d := \mathbb{R}[x_1, \ldots, x_n]_d$
$S = \{p_1, \ldots, p_s\} \subset \mathbb{R}^n.$

$\downarrow$ passing to algebraically closed fields

$R_{\mathbb{C},d} := \mathbb{C}[x_1, \ldots, x_n]_d$
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$R_{\mathbb{C},d}^{\text{homog}} := \mathbb{C}[x_0, x_1 \dots, x_n]_d$
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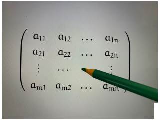
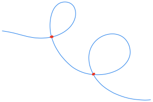




$$f(x_1, x_2) = \sum_{i,j} a_{i,j} x_1^i x_2^j$$

$$\downarrow$$

$$F(x_0, x_1, x_2) = \sum_{i,j} a_{i,j} x_0^{d-i-j} x_1^i x_2^j$$

- **Double point polynomial interpolation**: $\mathcal{L}_d(2S) = \{F \in R^{\text{homog}}_{\mathbb{C},d} : \frac{\partial}{\partial x_j} F(p_i) = 0, \forall i = 1, \ldots, s, \forall j = 0 \ldots, n\}$

Slogan

Double point polynomial interpolation problems	$\longleftrightarrow$	Dimensionality of secant varieties to varieties of rank one symmetric tensors

$$\sigma_t(X_d)$$

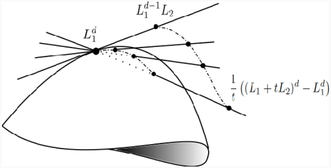
Tensors and polynomial interpolation

Slogan

Double point polynomial interpolation problems $\longleftrightarrow$ Dimensionality of secant varieties to varieties of rank one symmetric tensors

$$\sigma_t(X_d)$$

- Symmetric tensors:
  e.g. $\mathrm{Sym}^2(\mathbb{C}^n) = \{x \otimes y - y \otimes x : x, y \in \mathbb{C}^n\} \subset \mathbb{C}^n \otimes \mathbb{C}^n$
  $\mathrm{Sym}^d(\mathbb{C}^n) = R^{homog}_{\mathbb{C},d}$ i.e. degree-d polynomials
- $X_d \subseteq \mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^n))$ the Variety of rank-1 symmetric tensors
- $\sigma_t(X_d)$ is the t-secant variety to X_d



Question. How do we go from groups to tensors? Or the other way around?

Question. Why?

Tensors in finite group theory

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- **Computations** (isomorphism testing, automorphisms, ...)
- **Enumeration** (number of groups with given properties, number of groups with a shared quotient, ...)
- **Global techniques for local objects** (groups as points of a *scheme*).

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Finite groups are understood via their simple composition factors and via their Sylow p-subgroups. The first being classified, we look into the second.

Let p be a prime and $G_p = \langle g_1, g_2, g_3, h_1, h_2, h_3, z_1, z_2, z_3 \mid \text{relations} \rangle$ where:

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- z_1, z_2, z_3 central,
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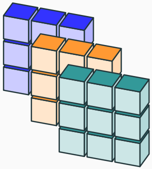
$$M = \begin{pmatrix} y_1 & y_2 & 0 \\ y_2 & -y_3 & y_1 \\ 0 & y_1 & -y_3 \end{pmatrix}$$



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$$\det(M) = y_1 y_3^2 - y_1^3 + y_3 y_2^2$$

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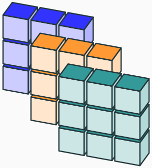

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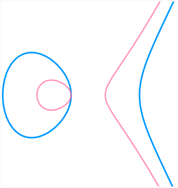
$$\det(M) = y_1 y_3^2 - y_1^3 + y_3 y_2^2$$

...and G_p is a *p-realization* of an object defined over $\mathbb{Z}$! ("**Globalization**")

Setting $\det(M) = 0$ we get a curve E in $\mathbb{P}^2$!



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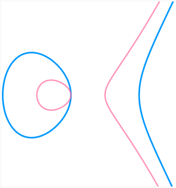
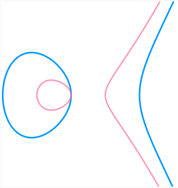


Question. What if I had another curve in the plane?

Question. Do I have a p-group associated to this curve?

Setting $\det(M) = 0$ we get a curve E in $\mathbb{P}^2$!



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Tensors in finite group theory

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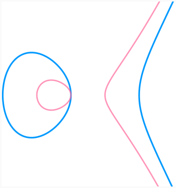

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Question. What are the isomorphisms of G_p? (**Enumeration**)

$$|\mathrm{Aut}(G_p)| = (\text{polynomial in } p) \times (\text{non-quasipolynomial}).$$

Let $X_1, \ldots, X_d$ discrete random with states $X_i \in [n_i] = \{1, \ldots, n_i\}$. Then we consider the tensor of **joint probabilities**: $T_{x_1,\ldots,x_d} = P(X_1 = x_1, \ldots, X_d = x_d)$.

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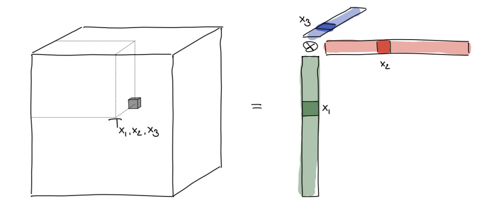
If we assume the variables to be **independent** then

$$P(X_1 = x_1, \dots, X_d = x_d) = P(X_1 = x_1) \cdots P(X_d = x_d),$$

namely, if we let $v_i = (P(X_i = 1), \dots, P(X_i = n_i))$,

$$T = v_1 \otimes v_2 \otimes \cdots \otimes v_d$$



Tensors in Statistical Models

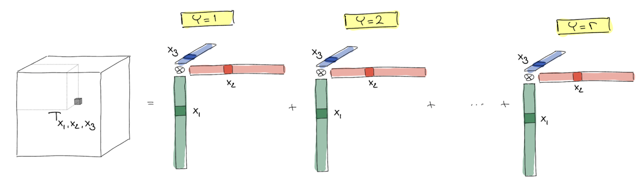
Let $X_1, \ldots, X_d$ discrete random with states $X_i \in [n_i] = \{1, \ldots, n_i\}$. Then we consider the tensor of **joint probabilities**: $T_{x_1,\ldots,x_d} = P(X_1 = x_1, \ldots, X_d = x_d)$.

If we assume the variables to be **independent conditionally to** $Y \in [r]$ then

$$
\begin{aligned}
P(X_1 = x_1, \ldots, X_d = x_d) &= \sum_{y=1}^{r} P(Y = y) P(X_1 = x_1, \ldots, X_d = x_d | Y = y) \\
&= \sum_{y=1}^{r} P(Y = y) P(X_1 = x_1 | Y = y) \cdots P(X_d = x_d | Y = y).
\end{aligned}
$$

namely, if we let $v_i^{(y)} = (P(X_i = 1 | Y = y), \ldots, P(X_i = n_i | Y = y))$, then

$$T = \sum_{y=1}^{r} \lambda_y v_1^{(y)} \otimes v_2^{(y)} \otimes \cdots \otimes v_d^{(y)}$$



Algebraic statistical model:

statistical model which depends *polynomially* on its parameters; namely

$$\text{a polynomial map} \quad \varphi : \mathcal{P} \longrightarrow \mathcal{M} \subset \Delta_N \subset \mathbb{R}^N$$

$$\mathcal{P} = \text{parameter space} \quad \mathcal{M} = \varphi(\mathcal{P}) \subset \Delta_N = \left\{ (p_0, \ldots, p_N) \in \mathbb{R}^N \ : \ {\scriptstyle p_0 + \ldots + p_N = 1 \atop p_i \geq 0} \right\}$$

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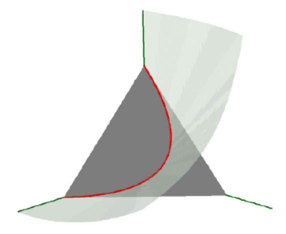
Example: Two throws of a biased coin C**.** Let $Z \in [2]$ count the number of heads.

If $P(C = H) = a$ and $p_i = P(Z = i)$, then

$$a \mapsto (p_0, p_1, p_2) = ((1-a)^2, 2a(1-a), a^2).$$

Then, $\mathcal{M} = \{(p_0, p_1, p_2) \in \Delta_2 \ : \ p_1^2 - 4p_0p_2 = 0\}$.



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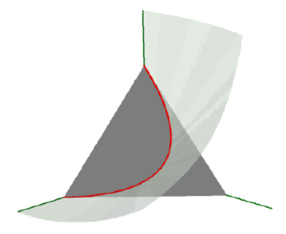

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Questions.

- What is the *dimension* of $\mathcal{M}$?
- What are the defining *equations* and *inequalities* of $\mathcal{M}$?
- Given $y \in \mathcal{M}$, how $\varphi^{-1}(y)$ look like?
  Is the model *identifiable*, i.e., the general fiber $\varphi^{-1}(y)$ is a singleton?

Not only discrete random variables and their joint probabilities can be approached by means of tensors and algebraic statistics.

Gaussian Models

Given a density function $f : \mathbb{R}^n \to \mathbb{R}$ for a random vector $X = (X_1, \ldots, X_n)$, its **moments** are

$$m_{i_1,\ldots,i_n} = \int_{\mathbb{R}^n} x_1^{i_1} \cdots x_n^{i_n} f(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

Example. For $n = 1$, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$m_1 = \mu, \;\; m_2 = \mu^2 + \sigma^2, \;\; m_3 = \mu^3 + 3\mu\sigma^2, \;\; m_4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4, \ldots$$

In general, moments of Gaussian models are polynomials in the parameters!

Tensors in Statistical Models

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Gaussian Models

Given $X_i \sim \mathcal{N}(\mu_i, \Sigma)$, $i = 1, \ldots, m$, and their *mixture* $Y = \lambda_1 X_1 + \ldots + \lambda_m X_m$, with $\lambda_1 + \ldots + \lambda_m = 1$, then

$$\varphi_d : (\lambda_1, \ldots, \lambda_m, \ldots, \mu_i, \Sigma_i, \ldots) \mapsto (m_{d0\cdots 0}, m_{d-1,1,0\cdots 0}, \ldots, m_{00\cdots d}).$$

This defines the **degree-d moment variety** of **mixture of Gaussian models**.

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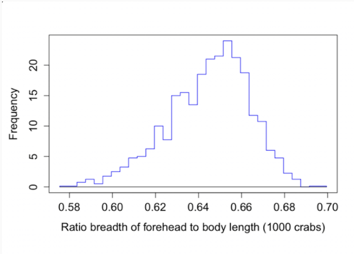
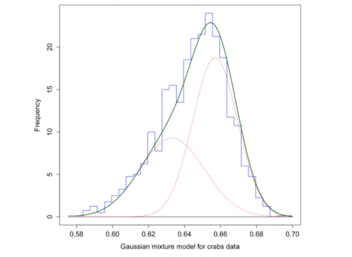
Method of moments (Pearson's crabs, 1903)



$\Longrightarrow$



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III. *Contributions to the Mathematical Theory of Evolution.*

By Karl Pearson, *University College, London.*

Communicated by Professor Henrici, *F.R.S.*

Received October 18,—Read November 16, 1893.

$$z_1 + z_2 = 1 \qquad (8).$$

$$\gamma_1 z_1 + \gamma_2 z_2 = 0 \qquad (9).$$

$$\gamma_1^2 z_1 (1 + u_1^2) + \gamma_2^2 z_2 (1 + u_2^2) = \mu_2 \qquad (10).$$

$$\gamma_1^3 z_1 (1 + 3u_1^2) + \gamma_2^3 z_2 (1 + 3u_2^2) = \mu_3 \qquad (11).$$

$$\gamma_1^4 z_1 (1 + 6u_1^2 + 3u_1^4) + \gamma_2^4 z_2 (1 + 6u_2^2 + 3u_2^4) = \mu_4 \qquad (12).$$

$$\gamma_1^5 z_1 (1 + 10u_1^2 + 15u_1^4) + \gamma_2^5 z_2 (1 + 10u_2^2 + 15u_2^4) = \mu_5 \qquad (13).$$

$$24p_2^9 - 28\lambda_4 p_2^7 + 36\mu_3^2 p_2^6 - (24\mu_3\lambda_5 - 10\lambda_4^2)\, p_2^5 - (148\mu_3^2\lambda_4 + 2\lambda_5^2)\, p_2^4$$
$$+ (288\mu_3^4 - 12\lambda_4\lambda_5\mu_3 - \lambda_4^3) p_2^3 + (24\mu_3^3\lambda_5 - 7\mu_3^2\lambda_4^2) p_2^2 + 32\mu_3^4\lambda_4 p_2 - 24\mu_3^6 = 0 \;. \qquad (29).$$

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Note:
$\mathrm{im}(\varphi_d)$ can be looked inside the space of degree-d multivariate polynomials

$$\bar{m}_d = \sum_{i_1,\ldots,i_n} m_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

If $\ell_i = \mu_i \cdot (x_1, \ldots, x_n)^T$ and $q_i = (x_1, \ldots, x_n)\Sigma_i(x_1, \ldots, x_n)^T$, then

$$\bar{m}_1 = \lambda_1\ell_1 + \ldots + \lambda_m\ell_m, \quad \bar{m}_2 = \lambda_1(\ell_1^2 + q_1) + \ldots + \lambda_m(\ell_m^2 + q_m)$$
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Tensors under group actions

Main aim: study geometric properties of secant varieties, e.g.

- identifiability of points = uniqueness in recovering data ;
- singularity of points = unfeasibility in computations .

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The group $\mathrm{GL}_n(\mathbb{C})$ acts on $\bigwedge^2 \mathbb{C}^n$ and leaves each $\sigma_r(\mathrm{Gr}_{2,n})$ invariant.

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$\Longrightarrow$ Enough to check them only on representatives of $\mathrm{GL}_n(\mathbb{C})$–orbits.

More in general:

Let G be a "nice" group. $= \mathrm{GL}_n(\mathbb{C})$
Let V^G be a vector space on which G acts without invariant proper vector subspaces. $= \bigwedge^d \mathbb{C}^n$ space of skew-symmetric tensors in $\bigwedge^d \mathbb{C}^n$
Let $v_0 \in V^G$ be a "special" vector. $= e_1 \wedge \ldots \wedge e_d$

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Bob: For instance, you find them in Quantum Physics ($\mathrm{Gr}_{d,n}$ as "simple" fermions), in Quantum Chemistry ($\mathrm{Gr}_{d,n}$ as quantum states of d electrons in n orbitals), in Quantum Information ("isotropic" Grassmannians parametrize abelian groups of observables in the Clifford group), ...

Tensors are everywhere... also in quantum computing!

Tensors and Quantum Computing

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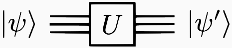
What is a quantum algorithm



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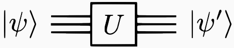

Two ingredients: Qubit system + Operations on them

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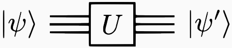

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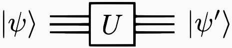

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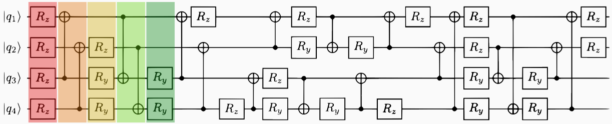
Goal: Find the best **quantum gate decomposition** $U = U_L \ldots U_2 \cdot U_1$



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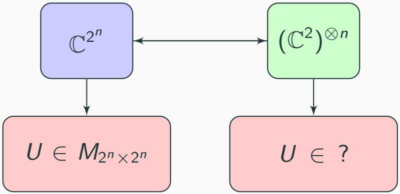

Entangling gates are **quasi-multilinear** maps

$$|\psi'\rangle = \sum_{k=1}^{s} (M_k^1, \ldots, M_k^n) \cdot |\psi\rangle$$

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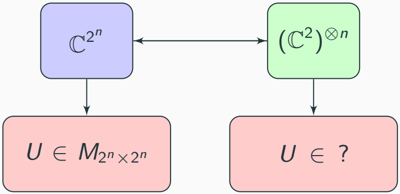
What if we change our point of view?

Two ingredients: Qubit system + Operations on them

- **Qubit states** as order-n tensors $|\psi\rangle = \sum_{i=1}^{r} \alpha_i (v_i^1 \otimes \ldots \otimes v_i^n) \in (\mathbb{C}^2)^{\otimes n}$
- **Quantum gates** as operations on tensors

$$U : (\mathbb{C}^2)^{\otimes n} \ni |\psi\rangle \longmapsto |\psi'\rangle \in (\mathbb{C}^2)^{\otimes n}$$



Entangling gates are **quasi-multilinear** maps

$$|\psi'\rangle = \sum_{k=1}^{s} (M_k^1, \ldots, M_k^n) \cdot |\psi\rangle$$

Goal: Find the best quantum gate decomposition can be viewed as find the **rank decomposition** of U

$$U = \sum_{k=1}^{s} M_k^1 \otimes \cdots \otimes M_k^n$$

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