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Why do we like to shatter tensors?

TensorDay 2024 - TensorDec @ UniTN

Daniele Taufer
KU Leuven
December 9, 2024.

fwo

Master degree

UNIVERSITÄT DUISBURG ESSEN

Offen im Denken

Algebra, Geometry and Number Theory

ALGANT Master Program

Famous for...

Master degree

UNIVERSITÄT DUISBURG ESSEN

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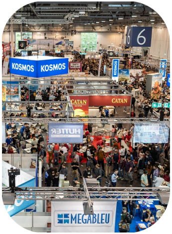


thyssenkrupp

Master degree

Famous for...

Master degree

Universiteit Leiden

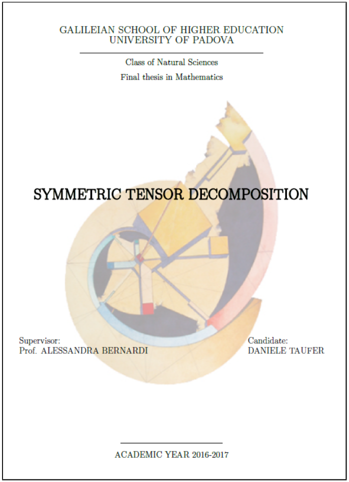
Algebra, Geometry and Number Theory
ALGANT Master Program

My Ph.D.

- Elliptic curves and loops
- ECC and DLP-based protocols
- First-degree prime ideals
- Symmetric tensor decomposition

Supervisor: Massimiliano Sala

My Ph.D. - Symmetric tensors


GALILEIAN SCHOOL OF HIGHER EDUCATION
UNIVERSITY OF PADOVA

Class of Natural Sciences

Final thesis in Mathematics

SYMMETRIC TENSOR DECOMPOSITION

Supervisor:
Prof. ALESSANDRA BERNARDI

Candidate:
DANIELE TAUFER

ACADEMIC YEAR 2016-2017

Symmetric Tensor Decomposition

Daniele Taufer
supervised by Alessandra Bernardi

University of Trento

November 21, 2017

D. Taufer, A. Bernardi (UniTN) Symmetric Tensor Decomposition November 21, 2017 1 / 21

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First postdoc

CISPA
HELMHOLTZ CENTER FOR
INFORMATION SECURITY

founded by ERC-669891
(Antoine Joux)

Saarbrücken

Famous for...

Second postdoc (current)

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Grant: 12ZZC23N

Leuven

Famous for...

Second postdoc (current)

Grant: 12ZZC23N

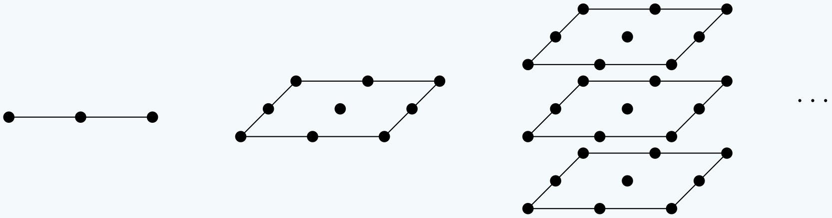
Leuven

Famous for... STELLA ARTOIS $\subset$ ABInBev

Introducing: tensors*

An order-d tensor over V is an element of $\underbrace{V \otimes V \otimes \cdots \otimes V}_{d \text{ times}}$.

Example $(\dim V = 3)$



V $\qquad$ $V \otimes V$ $\qquad$ $V \otimes V \otimes V$ $\qquad$ $\cdots$

Simple tensors

Linearity

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w.$$

Rank-1 tensors*

$$\begin{aligned}(e_1 + 2e_2)^{\otimes 3} &= (e_1 + 2e_2) \otimes (e_1 + 2e_2) \otimes (e_1 + 2e_2) \\ &= e_{1,1,1} + 2e_{1,1,2} + 2e_{1,2,1} + 2e_{2,1,1} \\ &\quad + 4e_{1,2,2} + 4e_{2,1,2} + 4e_{2,2,1} + 8e_{2,2,2},\end{aligned}$$

where

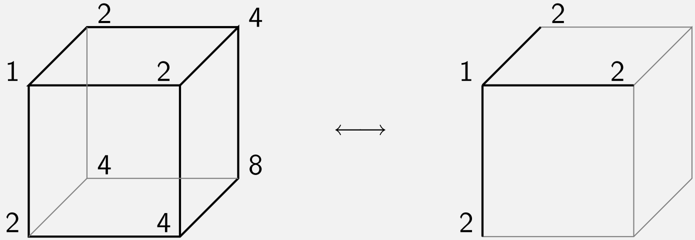
$$e_{i,j,k} = e_i \otimes e_j \otimes e_k.$$

Simple tensors

Linearity

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w.$$

Rank-1 tensors*



Waring decomposition problem*

$$(e_1 + 2e_2)^{\otimes 3}$$

Expand ?

$$e_{1,1,1} + 2e_{1,1,2} + 2e_{1,2,1} + 2e_{2,1,1} + 4e_{1,2,2} + 4e_{2,1,2} + 4e_{2,2,1} + 8e_{2,2,2}$$

Waring decomposition problem*

$$(e_1 + 2e_2)^{\otimes 3}$$

Expand ↓ ↑ ?

$$e_{1,1,1} + 2e_{1,1,2} + 2e_{1,2,1} + 2e_{2,1,1} + 4e_{1,2,2} + 4e_{2,1,2} + 4e_{2,2,1} + 8e_{2,2,2}$$

Waring decomposition

Writing a tensor as a sum of the **minimal** number of rank-1 tensors. Such a minimal number is referred to as its *Waring rank*.

Symmetric version

Symmetric tensors

A tensor is called *symmetric* if it is invariant under (basis) index permutations.

$$e_{1,1,1} + 2e_{1,1,2} + 2e_{1,2,1} + 2e_{2,1,1} + 4e_{1,2,2} + 4e_{2,1,2} + 4e_{2,2,1} + 8e_{2,2,2}$$

Same coefficients: $e_{i,j,k},\ e_{i,k,j},\ e_{j,i,k},\ e_{j,k,i},\ e_{k,i,j},\ e_{k,j,i}$

Symmetric version

Symmetric tensors

A tensor is called *symmetric* if it is invariant under (basis) index permutations.

$$\underbrace{e_{1,1,1}}_{x_1^3} + \underbrace{2e_{1,1,2} + 2e_{1,2,1} + 2e_{2,1,1}}_{6x_1^2x_2} + \underbrace{4e_{1,2,2} + 4e_{2,1,2} + 4e_{2,2,1}}_{12x_1x_2^2} + \underbrace{8e_{2,2,2}}_{8x_2^3}$$

Same coefficients: $\underbrace{e_{i,j,k},\ e_{i,k,j},\ e_{j,i,k},\ e_{j,k,i},\ e_{k,i,j},\ e_{k,j,i}}_{x_ix_jx_k}$

Symmetric version

Symmetric Waring decomposition problem

Write a homogeneous degree-d polynomial $F \in \mathbb{k}[x_0, \ldots, x_n]$ as a sum of the minimal number of d-powers of linear forms.

Theorem [Alexander–Hirschowitz]

If F is *generic*, its expected rank is $\left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$, except for:

(d, n)	rk(F)	Instead of
$(2, n)$	n+1	$\lceil \frac{n+2}{2} \rceil$
$(3, 4)$	8	7
$(4, 2)$	6	5
$(4, 3)$	10	9
$(4, 4)$	15	14

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Apolarity

Annihilator

Let F be an homogeneous polynomial. Its *annihilator* is the homogeneous ideal

$$\mathrm{Ann}(F) = \{G \in \Bbbk[\partial_0, \dots, \partial_n] \mid G \circ F = 0\}.$$

Apolarity Lemma

The following are equivalent:

- $F = L_1^d + \cdots + L_s^d$,
- there is an ideal $I \subset \mathrm{Ann}(F)$ defining s simple points in $\mathbb{P}^n(\Bbbk)$.

Why do we care?

Top-level reason

- A natural, insanely difficult, algebraic problem!

High-level reasons

1. Optimization and efficiency
2. Revealing hidden information
3. Measuring complexity

Different flavors

Commutative and computational algebra, algebraic geometry, combinatorics, numerical analysis, ...

1. Optimization and efficiency

Example: storage cost

$$F = 3x_0^4 + 8x_0^3x_1 + 4x_0^3x_2 + 12x_0^2x_1^2 + 24x_0^2x_1x_2 + 30x_0^2x_2^2 + 8x_0x_1^3 + 24x_0x_1^2x_2 + 48x_0x_1x_2^2 + 28x_0x_2^3 + 2x_1^4 + 8x_1^3x_2 + 24x_1^2x_2^2 + 32x_1x_2^3 + 17x_2^4.$$

We need $\binom{4+2}{2} =$ **15** coefficients to store its monomial representation. But if one knows that

$$F = (x_0 + x_1)^4 + (x_0 - x_2)^4 + (x_0 + x_1 + 2x_2)^4,$$

then **9** coefficients are enough.

Sometimes we are happy with low-rank approximations!

2. Revealing hidden information

If a tensor arises from a physical system, it is probably not random.

The pieces of a minimal decomposition usually represent intrinsic properties/objects that play a special role in the considered system.

2. Revealing hidden information

Example: roots of univariate cubics

- Let $f \in \mathbb{k}[x]$ be a random degree-3 polynomial.
- Homogenize it: $F \in \mathbb{k}[x, y]$.
- By A-H with $(d, n) = (3, 1)$, we know that the expected rank is 2:

$$F = L_1^3 + L_2^3.$$

- The solutions to $F = 0$ are

$$L_1 = -\sqrt[3]{1} \cdot L_2.$$

- Solve in x with $y = 1$: get the roots!

3. Measuring complexity

Multiplying (linear) polynomials

$$(a_0 + a_1 x)(b_0 + b_1 x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + a_1 b_1 x^2.$$

We can do it with 4 $\mathbb{k}$-multiplications. Can we do better?

$$F = a_0 b_0 c_0 + (a_0 b_1 + a_1 b_0) c_1 + a_1 b_1 c_2.$$

3. Measuring complexity

Multiplying (linear) polynomials

$$(a_0 + a_1 x)(b_0 + b_1 x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + a_1 b_1 x^2.$$

We can do it with 4 $\mathbb{k}$-multiplications. Can we do better?

$$\begin{aligned} F &= a_0 b_0 c_0 + (a_0 b_1 + a_1 b_0) c_1 + a_1 b_1 c_2. \\ &= a_0 b_0 (c_0 - c_1) + (a_0 + a_1)(b_0 + b_1) c_1 + a_1 b_1 (-c_1 + c_2). \end{aligned}$$

↪ "Karatsuba decomposition"

3. Measuring complexity

Multiplying (linear) polynomials

$$(a_0 + a_1 x)(b_0 + b_1 x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + a_1 b_1 x^2.$$

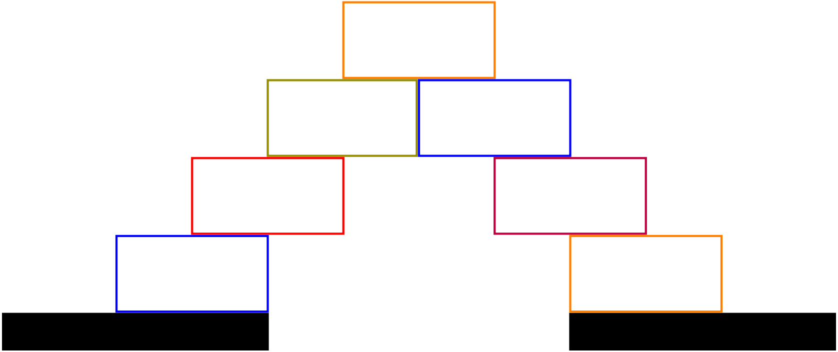
We can do it with 4 $\Bbbk$-multiplications. Can we do better? YES!

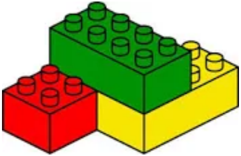
With 3 $\Bbbk$-multiplications

$$= a_0 b_0 + \big(-a_0 b_0 + (a_0 + a_1)(b_0 + b_1) - a_1 b_1\big)x + a_1 b_1 x^2$$

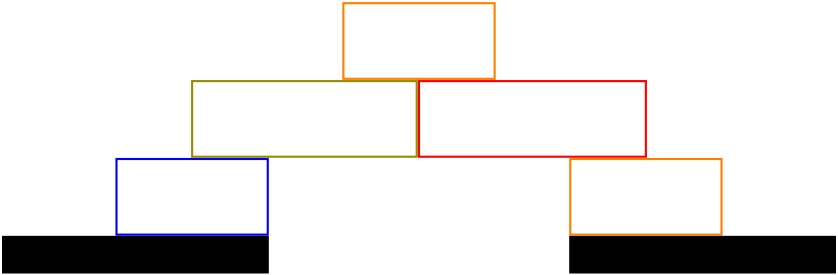
$\rightsquigarrow$ We should look at more sophisticated decompositions!

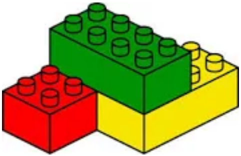
Generalized additive decomposition



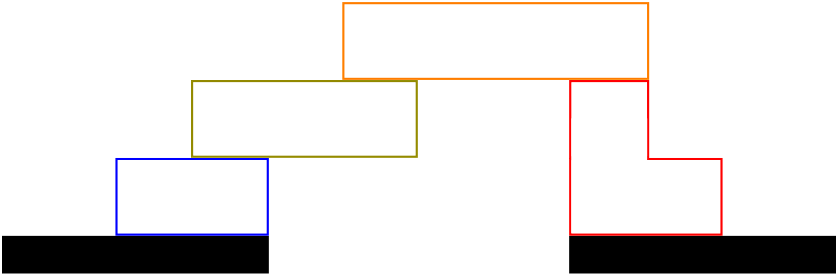


Generalized additive decomposition





Generalized additive decomposition



$$F = \sum_{i=1}^{s} L_i^{d-k_i} G_i.$$

Thanks for your attention!

daniele.taufer@gmail.com

A. Bernardi, E. Carlini, M. V. Catalisano, A. Gimigliano, A. Oneto, *The Hitchhiker Guide to: Secant Varieties and Tensor Decomposition*, Mathematics 6(12), 314, 2018.