CNR-INO
ISTITUTO NAZIONALE DI OTTICA
CONSIGLIO NAZIONALE DELLE RICERCHE

BEC
BOSE EINSTEIN CONDENSATION

UNIVERSITÀ DI TRENTO

Tensors: the physicist's all-rounder

Iacopo Carusotto

Pitaevskii BEC Center, INO-CNR and Dipartimento di Fisica, Università di Trento
I-38123 Trento, Italy

PROVINCIA AUTONOMA DI TRENTO

Q@TN

QST

Funded by the European Union
NextGenerationEU

Earliest Known Uses of Some of the Words of Mathematics (T)

TENSOR was one of the family of terms introduced by William Rowan Hamilton (1805-1865) in his study of QUATERNIONS. VECTOR and SCALAR and VERSOR were among the others. The tensor is for quaternions what the MODULUS is for complex numbers. The term derives from the Latin *tendĕre* to stretch.

In 1846 Hamilton wrote in *The London, Edinburgh, and Dublin Philosophical Magazine* XXIX. 27:

> *Since the square of a scalar is always positive, while the square of a vector is always negative, the algebraical excess of the former over the latter square is always a positive number; if then we make* $(TQ)^2 = (SQ)^2 — (VQ)^2$*, and if we suppose TQ to be always a real and positive or absolute number, which we may call the tensor of the quaternion Q, we shall not thereby diminish the generality of that quaternion. This tensor is what was called in former articles the modulus.*

The passage is reproduced in Section 19 of "On Quaternions". This 'article' is a compilation of 18 short papers published in the *Philosophical Magazine* between 1844 and 1850 made by the editors of Hamilton's *Mathematical Papers*. The editors concatenated them to form a seamless whole, with no indication as to how the material was distributed into the individual papers.

Tensor in Hamilton's sense is no longer used.

[Information for this article was provided by David Wilkins and Julio González Cabillón.]

MT MacTutor

HOME BIOGRAPHIES HISTORY TOPICS MAP CURVES SEARCH

Earliest Known Uses of Some of the Words of Mathematics (T)

TENSOR, TENSOR ANALYSIS, TENSOR CALCULUS, etc. are 20th century terms associated with the ABSOLUTE DIFFERENTIAL CALCULUS developed by Ricci-Curbastro in the 1880s and -90s on the basis of earlier work by Riemann, Christoffel, Bianchi and others. See Kline ch. 37 "The Differential Geometry of Gauss and Riemann" and ch. 48 "Tensor Analysis and Differential Geometry."

Ricci's most influential publication was a substantial article written with his former student Levi-Civita. The article by the two Italians was written in French and appeared in the leading German mathematical journal: "Méthodes de calcul différentiel absolu et leurs applications," *Mathematische Annalen,* **54** (1901), p. 125-201. The word *tensor* does *not* appear: Ricci and Levi-Civita write about *systèmes*. *Tensor* is due to the well-known Göttingen physicist Woldemar Voigt (1850-1919), who used it in his *Die fundamentalen physikalischen Eigenschaften der Krystalle in elementarer Darstellung* of 1898 (*OED* and Julio González Cabillón).

Critically *tensor* was the term adopted by Einstein and Grossmann in their first publication on general relativity, *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation* (*Outline of a Generalized Theory of Relativity and of a Theory of Gravitation*) (1913). Einstein made the subject fashionable. MacTutor relates that on a visit to Princeton in 1921 he commented on the large audience his lecture attracted: "I never realised that so many Americans were interested in tensor analysis." See also MacTutor: General Relativity.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein eqs of gravity in General Relativity

Space-time metric $g_{\mu\nu}$; Ricci curvature $R_{\mu\nu}$ determined by stress-energy tensors $T_{\mu\nu}$

2. ^ Woldemar Voigt, *Die fundamentalen physikalischen Eigenschaften der Krystalle in elementarer Darstellung* [The fundamental physical properties of crystals in an elementary presentation] (Leipzig, Germany: Veit & Co., 1898), p. 20. From page 20: *"Wir wollen uns deshalb nur darauf stützen, dass Zustände der geschilderten Art bei Spannungen und Dehnungen nicht starrer Körper auftreten, und sie deshalb tensorielle, die für sie charakteristischen physikalischen Grössen aber Tensoren nennen."* (We therefore want [our presentation] to be based only on [the assumption that] conditions of the type described occur during stresses and strains of non-rigid bodies, and therefore call them "tensorial" but call the characteristic physical quantities for them "tensors".)

- Voigt effect in magneto-optics
- Voigt lineshape in spectroscopy
- Voigt notation of symmetric tensors

Stress tensor T^{ij}

Strain tensor E_{kl}

Generalized Hooke's law of elasticity

$$T^{ij} = C^{ijkl} E_{kl}$$

Elasticity tensor C^{ijkl} of given material

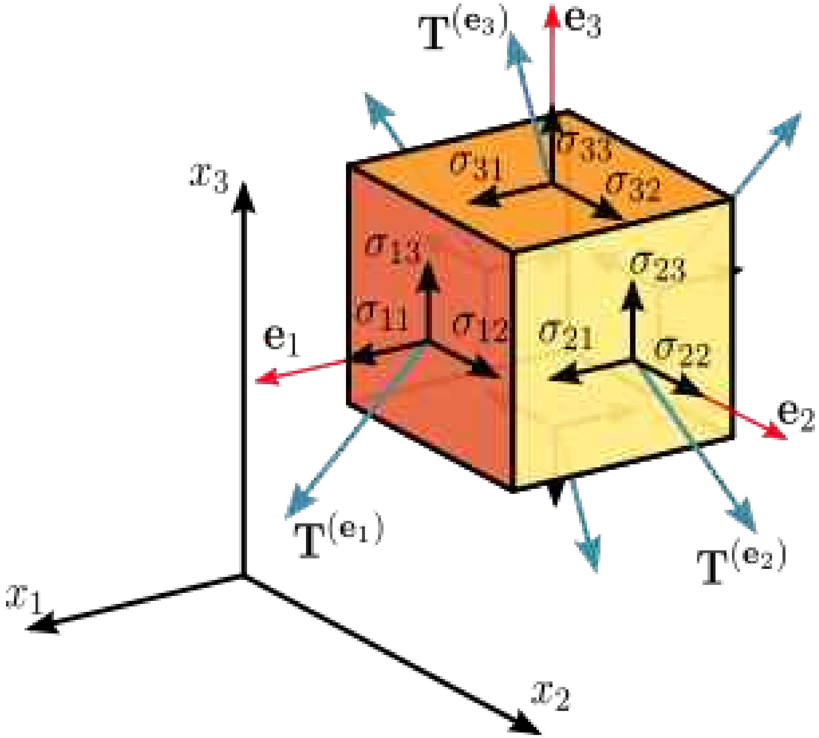
has $3^4 = 81$ components:

- How many really independent?
- Can one decompose it in simpler elements?

Woldemar Voigt

Woldemar Voigt, c. 1908

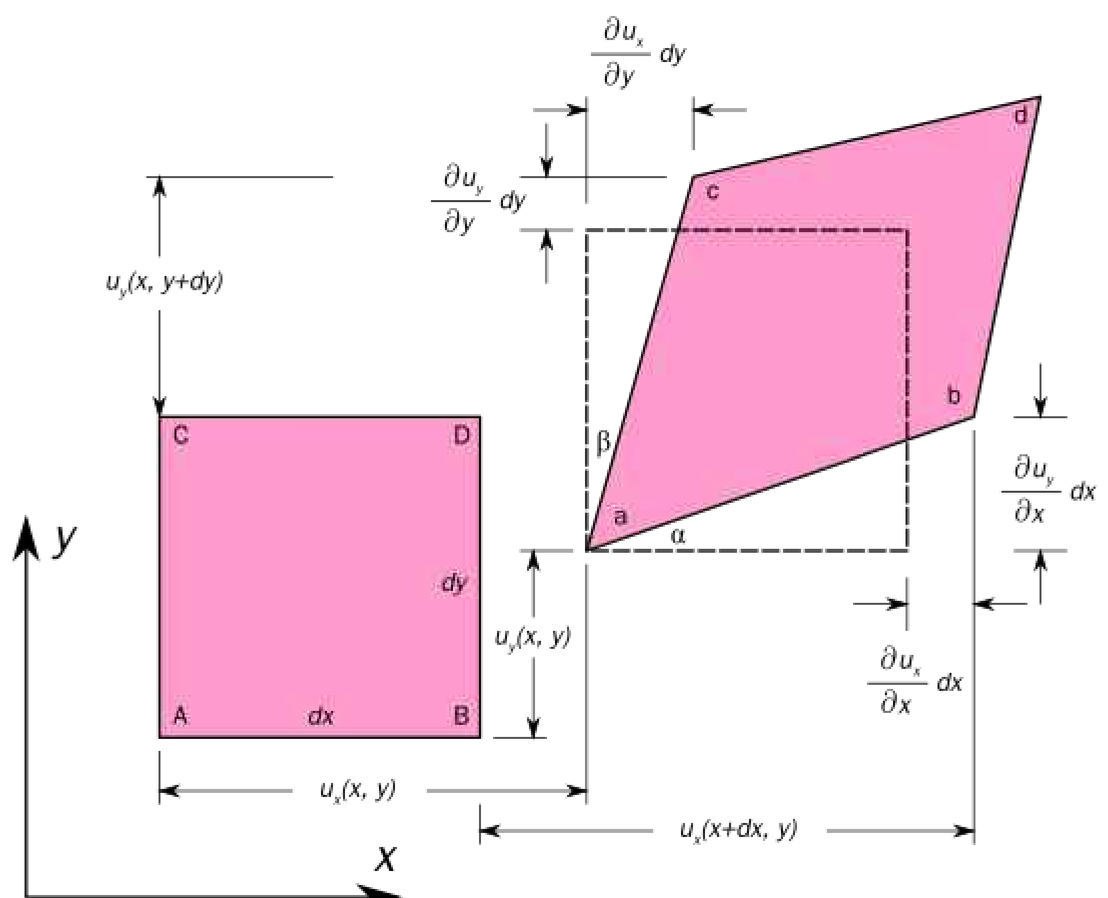
Born	2 September 1850 Leipzig, Kingdom of Saxony
Died	13 December 1919 (aged 69) Göttingen, Germany



Stress tensor T^{ij} determnes mechanical force $F^i = T^{ij} e_j$

Strain tensor describes deformation

$E_{kl} = \partial u_k / \partial r_l$



Tensors at work @ BEC Center

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS B: ATOMIC, MOLECULAR AND OPTICAL PHYSICS

J. Phys. B: At. Mol. Opt. Phys. **37** (2004) S115–S125 PII: S0953-4075(04)71248-1

Imaging of spinor gases

Iacopo Carusotto[1,2] and Erich J Mueller[3]

Light-atom interaction Hamiltonian:

$$H_{\text{int}} = \int \mathrm{d}\mathbf{x} \sum_{\alpha i j \beta} U_{\alpha i j \beta} \hat{\psi}^{\dagger}_{\alpha}(\mathbf{x}) \hat{E}^{\dagger}_{i}(\mathbf{x}) \hat{E}_{j}(\mathbf{x}) \hat{\psi}_{\beta}(\mathbf{x})$$

$\alpha,\beta \rightarrow$ internal atomic states [s representation of O(3)]

$i,j \rightarrow$ Cartesian indices [$l=1$ representation of O(3)]

$U_{\alpha ij\beta} \rightarrow$ big tensor with $[3(2s+1)]^2$ elements. Can we write it simply?

- Free-space → rotational symmetry. R(U) = U for any R
- $\psi^+ \otimes \psi = 2s \oplus \ldots \oplus 1 \oplus 0$ and $E^+ \otimes E = 2 \oplus 1 \oplus 0$ → 3 scalars (for s≥1)

$$H_{\text{int}} = \int \mathrm{d}\mathbf{x} \left[b_0 I(\mathbf{x}) \rho(\mathbf{x}) + b_1 \Sigma(\mathbf{x}) \cdot \mathbf{S}(\mathbf{x}) + b_2 \sum_{ij} \mathcal{N}_{ij}(\mathbf{x}) N_{ij}(\mathbf{x}) \right]$$

Refractive index

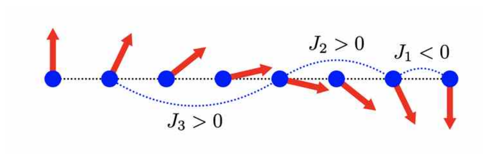
Polarization rotation

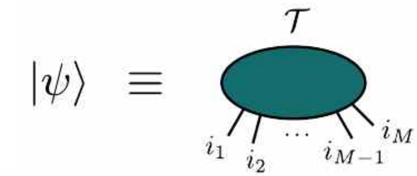
Birefringence

Tensor networks

General many-body system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \ldots \otimes \mathcal{H}_N$

e.g. a spin chain





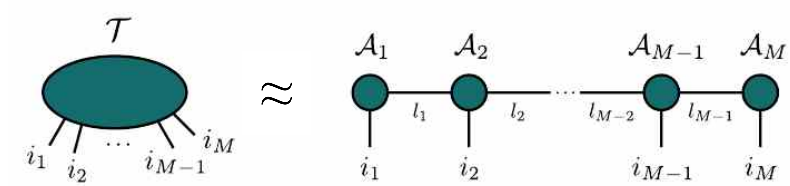
$$|\psi\rangle = \sum_{n_1,\ldots,n_M} \mathcal{T}_{n_1,\ldots,n_M} |n_1,\ldots,n_M\rangle$$

Physical dimensions

Dimension $d_{tot}=d_1\, d_2\, \ldots\, d_N$ scales exponentially with N

<u>Tensor network ansatz:</u>

- Recovers any state for suffic. large bond dim's b_j
- Good approx. of 'relevant' states in $\mathcal{H}$ for tractable b's
- Exact small b representation of some states, e.g. AKLT



$$\mathcal{T}_{n_1,\ldots,n_M} \approx \sum_{l_0,\ldots,l_M} \mathcal{A}_1{}^{n_1}_{l_0,l_1} \cdots \mathcal{A}_M{}^{n_M}_{l_{M-1},l_M}$$

Bond dimension

$l_j = 1 \ldots b_j$

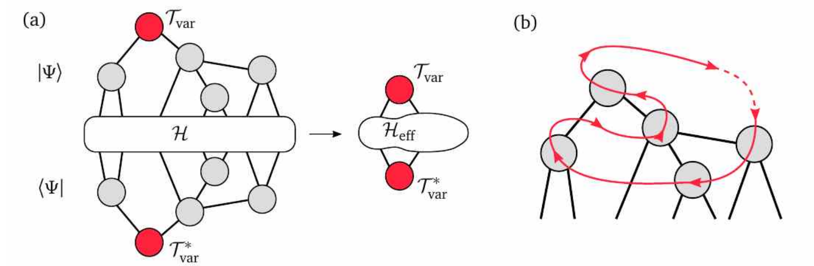
Variational approximation of ground state: minimization of E over TN's

Variational ground state on TNs

Most common application of TN's to physics:

- Complex network of tensors
- Evaluate expectation value of energy

- Minimize energy by acting on each tensor individually
- C. De Lazzari's PhD project → global minimization



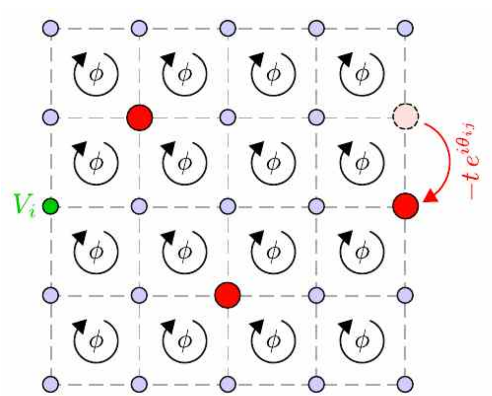
Example of application:

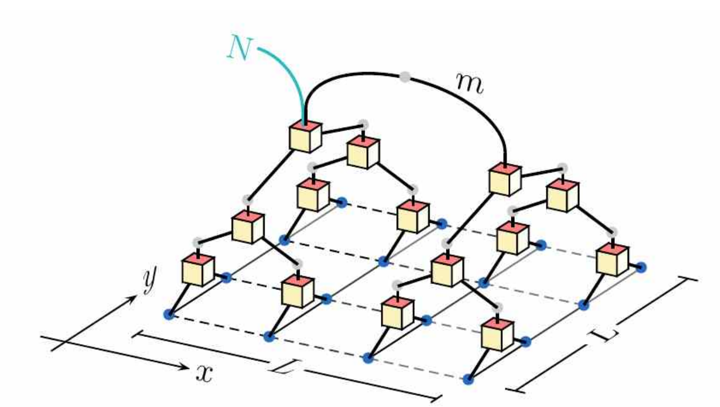
- Gas of quantum particles
- 2D lattice + B field
- Tree-tensor-network representation (makes calculations tractable)

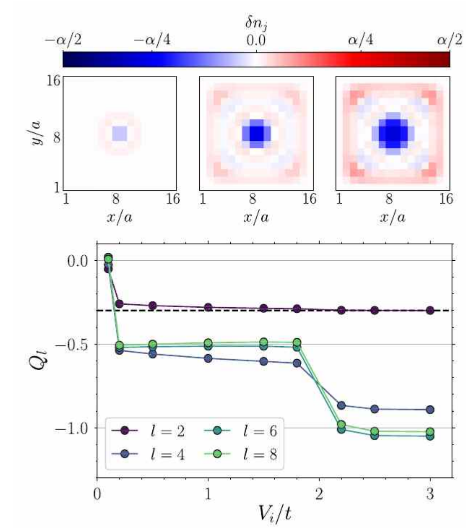
PHYSICAL REVIEW RESEARCH **2**, 013145 (2020)

Charge and statistics of lattice quasiholes from density measurements: A tree tensor network study

E. Macaluso,[1] T. Comparin,[1,2] R. O. Umucalılar,[3] M. Gerster,[4] S. Montangero,[5] M. Rizzi,[6,7] and I. Carusotto[1]

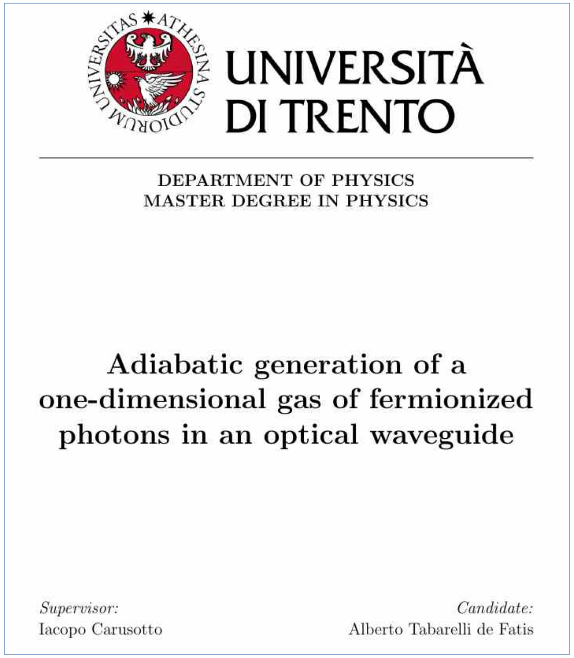






Experimental challenge: realize a gas of strongly interacting photons 1/2

Alberto Tabarelli De Fatis



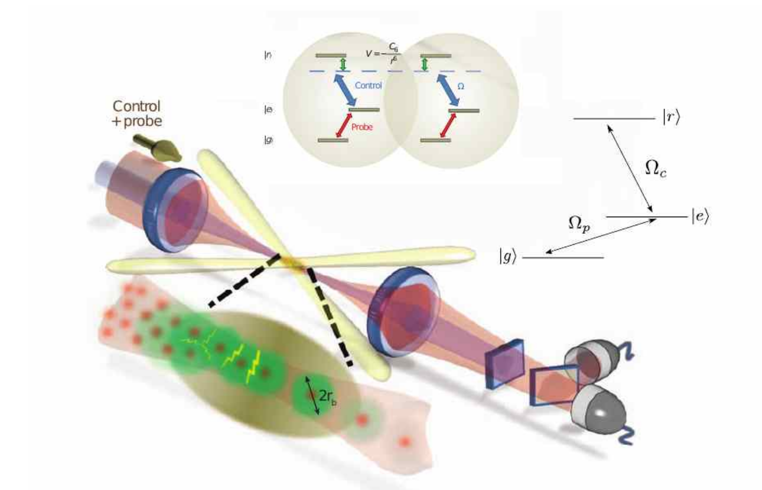
UNIVERSITÀ DI TRENTO

DEPARTMENT OF PHYSICS
MASTER DEGREE IN PHYSICS

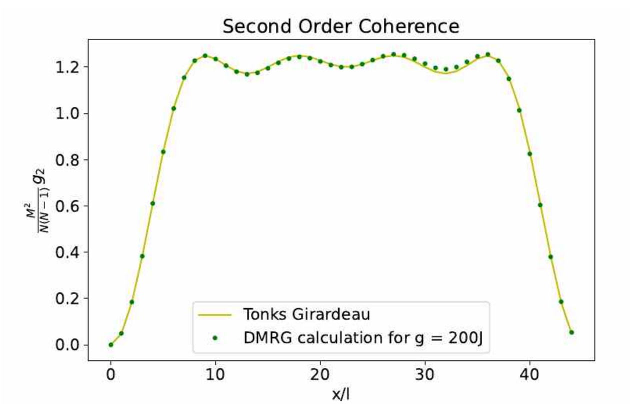
Adiabatic generation of a one-dimensional gas of fermionized photons in an optical waveguide

Supervisor: Iacopo Carusotto

Candidate: Alberto Tabarelli de Fatis

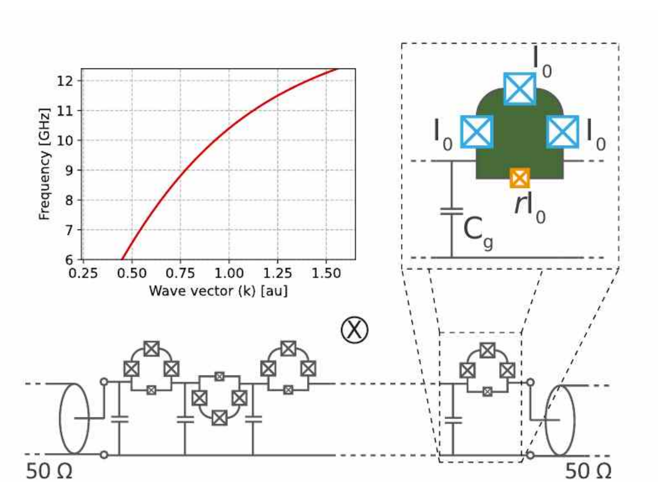


The MIT set-up (Peyronel et al. 2012)



TN calculation of ground state via energy minimization

Can the ground state be reached in experiments?



The set-up in construction @ FBK

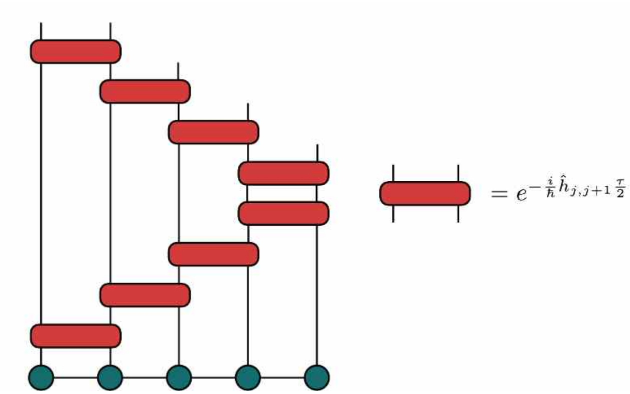
Experimental challenge: realize a gas of strongly interacting photons 2/2

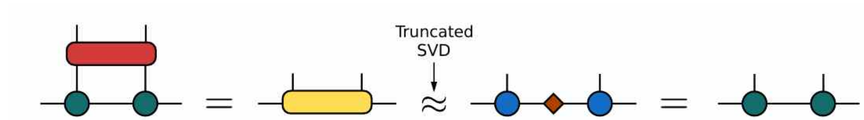
Hamiltonian evolution of quantum system → Schroedinger eq: $i\,\partial_t \psi = H\,\psi$

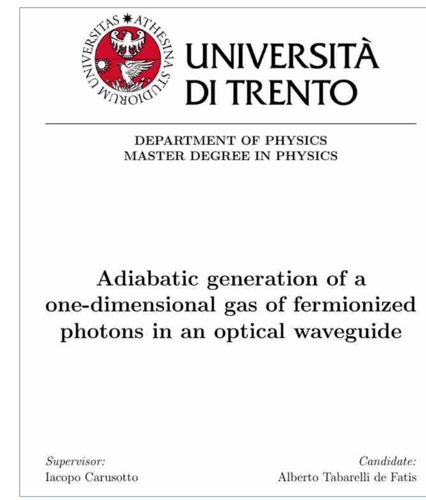
H = sum of (non-commuting) two-site terms

Within small time step Δt → $\exp(-i\,(H_1+H_2)\,t) \approx \exp(-i\,H_1\,t/2)\,\exp(-i\,H_2\,t)\,\exp(-i\,H_1\,t/2)$

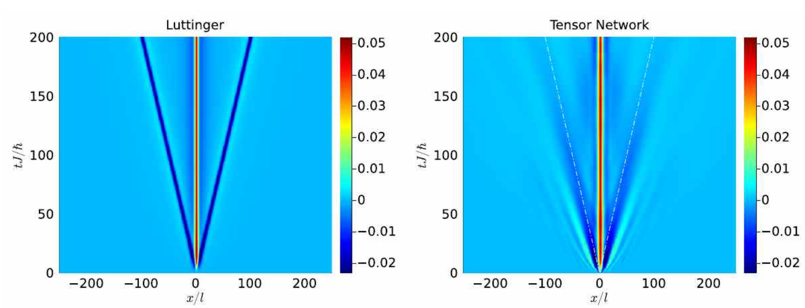
Two-site operators generate entanglement, to be then distilled via SVD

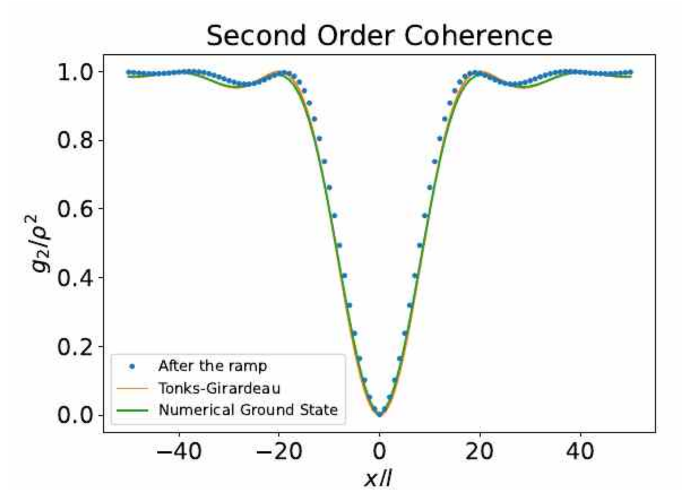






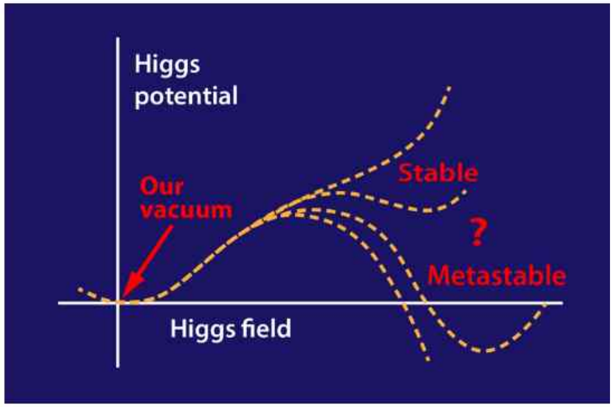
Example of time-evolution





Adiabatic evolution maintains ground state

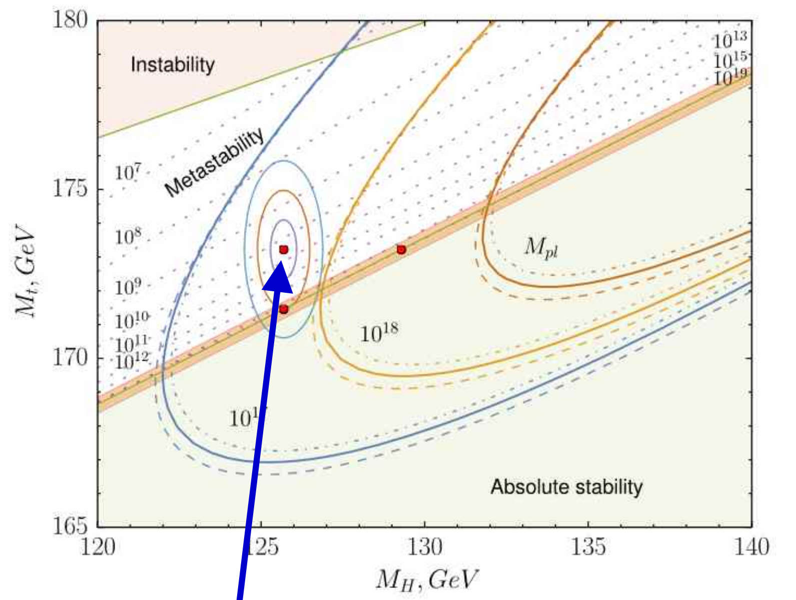
Could quantum mechanics destroy our Universe?



According to standard model
Higgs field may be in metastable state

Tunneling event towards global energy minimum
→ totally different physics and huge energy release
The END of our world !

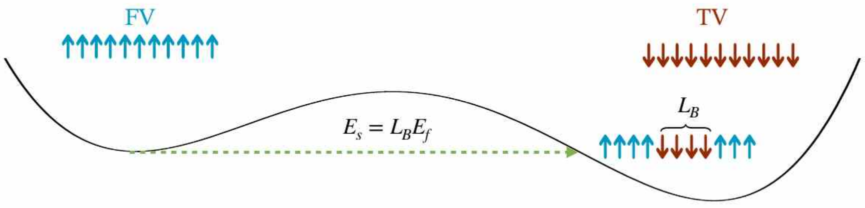
Existing calcul.→ very slow decay (~10^{65} yrs), but...



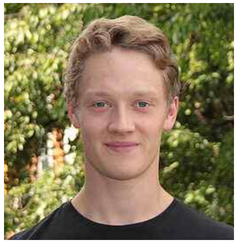
Our Universe is expected to be metastable
but not far from stability boundary
(many hypothesis and approximations behind calculation)

Estimated decay rate not so reliable:

- Instanton calcul. only tractable under serious approx
- Full calcul.needed to unravel physics on simplified model



$$H(h_l) = -J\left(\sum_i \sigma_i^x \sigma_{i+1}^x + h_T \sum_i \sigma_i^z + h_l \sum_i \sigma_i^x\right)$$



C. Johansen

Time-dependence
of spin-chain model
to be studied via TNs

TENORS DC10 – Geometry of tensor network varieties for quantum condensed matter physics

Dynamics on the TN manifold → approximates quantum dynamics of strongly correlated states

Example: b=1:

$$|\Psi_G\rangle = \bigotimes_{\mathbf{r}} \sum_n c_n(\mathbf{r})\, |n, \mathbf{r}\rangle$$

PHYSICAL REVIEW RESEARCH **2**, 033276 (2020)

Quantum fluctuations beyond the Gutzwiller approximation in the Bose-Hubbard model

Fabio Caleffi,[1,2,*] Massimo Capone,[1,3] Chiara Menotti,[2] Iacopo Carusotto,[2,4] and Alessio Recati[2,4,†]

- Effective Lagrangian

$$\begin{aligned}\mathfrak{L}[c, c^*] &= \langle \Psi_G|\, i\hbar\, \partial_t - \hat{H}\, |\Psi_G\rangle \\ &= \frac{i\hbar}{2} \sum_{\mathbf{r},n} [c_n^*(\mathbf{r})\dot{c}_n(\mathbf{r}) - \text{c.c.}] \\ &\quad + J \sum_{\langle \mathbf{r},\mathbf{s}\rangle} [\psi^*(\mathbf{r})\, \psi(\mathbf{s}) + \text{c.c.}] - \sum_{\mathbf{r},n} H_n |c_n(\mathbf{r})|^2\end{aligned}$$

→ Euler-Lagrange eqs for classical dynamics

- Quantize theory, e.g. via path-integral
- Small fluctuation $\hat{c}_n(\mathbf{r}) = \hat{A}(\mathbf{r})\, c_n^0 + \delta\hat{c}_n(\mathbf{r}).$ → Collective excitation modes

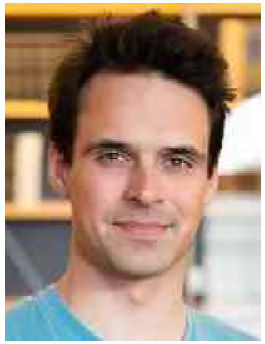
$$\hat{H}^{(2)} = \hbar \sum_{\alpha} \sum_{\mathbf{k}} \omega_{\alpha,\mathbf{k}}\, \hat{b}^\dagger_{\alpha,\mathbf{k}} \hat{b}_{\alpha,\mathbf{k}},$$

Otto's project: extend method to general b>1 TN's and apply it to complex states (e.g. AKLT, for which ground state has b=2 TN form)

Otto Schmidt

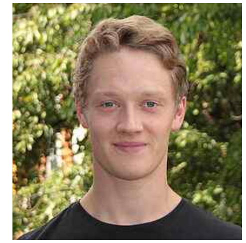
The team

Otto Schmidt
PhD student
TENORS project



Simon Telen
MPI MPS Leipzig
Otto's -cosupervisor

Alberto Tabarelli De Fatis
PhD student

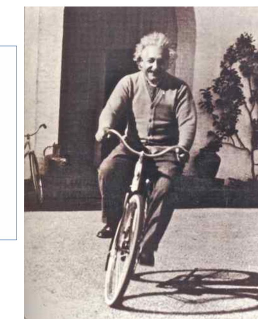


Christian Johansen
PostDoc

Alberto Biella
INO-CNR BEC

Alessandra Bernardi
DM-UniTN

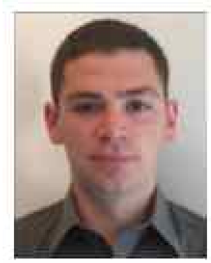
> I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot.
>
> Albert Einstein to Levi-Civita, 1917



Collaborators:

Matteo Rizzi
U. Cologne & Julich

Alumni:



Elia Macaluso
now risk manager
@ Cassa Centrale Banca

Claudia De Lazzari
Now R&D @ QTI

Interested in joining and/or collaborating? *iacopo.carusotto@ino.cnr.it*