Decoding tensors through a geometric lens

Pierpaola Santarsiero
pierpaola.santarsiero@unibo.it

University of Bologna, Italy

TensorDay 2024

Finanziato dall'Unione europea NextGenerationEU | Ministero dell'Università e della Ricerca | Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA | ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA

What is a tensor?

- it is a **multidimensional** table of numbers
- it is a **multilinear** map

What is a tensor?

- it is a **multidimensional** table of numbers
- it is a **multilinear** map
- it is a~~n **element**~~ **point** of the **space** $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \cong \mathbb{C}^{n^3}$

The field of tensor decomposition

A **tensor** T is an element of $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ that can be written in coordinates as:

$$T = \sum_{i,j,k} t_{i,j,k} e_i \otimes e_j \otimes e_k.$$

The field of tensor decomposition

A **tensor** T is an element of $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ that can be written in coordinates as:

$$T = \sum_{i,j,k} t_{i,j,k} e_i \otimes e_j \otimes e_k.$$

The field of tensor decomposition focuses on **minimizing** the number of summands needed in such decomposition of the tensor.

The field of tensor decomposition

A **tensor** T is an element of $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ that can be written in coordinates as:

$$T = \sum_{i,j,k} t_{i,j,k} e_i \otimes e_j \otimes e_k.$$

The field of tensor decomposition focuses on **minimizing** the number of summands needed in such decomposition of the tensor.

Example

$T = 3e_1 \otimes e_1 \otimes e_1 - 3e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 - e_1 \otimes e_2 \otimes e_2 + 6e_2 \otimes e_1 \otimes e_1 - 2e_2 \otimes e_1 \otimes e_2 - 2e_2 \otimes e_2 \otimes e_2 \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ **is** actually the tensor

$$T = (e_1 + 2e_2) \otimes (3e_1 + e_2) \otimes (e_1 - e_2).$$

Tensor rank

An **elementary tensor** is a tensor of the form $u \otimes v \otimes w$ for some $u, v, w \in \mathbb{C}^n$.

Tensor rank

An **elementary tensor** is a tensor of the form $u \otimes v \otimes w$ for some $u, v, w \in \mathbb{C}^n$.

The **rank** of a tensor T is the smallest r such that

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i, \quad u_i, v_i, w_i \in \mathbb{C}^n.$$

Tensor rank

An **elementary tensor** is a tensor of the form $u \otimes v \otimes w$ for some $u, v, w \in \mathbb{C}^n$.

The **rank** of a tensor T is the smallest r such that

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i, \quad u_i, v_i, w_i \in \mathbb{C}^n.$$

Elementary tensors are rank-1 tensors.

The variety of elementary tensors

Elementary tensors in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ are parametrized by the **Segre variety**:
$\mathrm{Seg}(\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n) :=$

$$\begin{aligned} &\{T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \,|\, T = a \otimes b \otimes c, \text{ for some } a, b, c \in \mathbb{C}^n\} \\ =&\{T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \,|\, \mathrm{rk}(T) = 1\}. \end{aligned}$$

The variety of elementary tensors

Elementary tensors in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ are parametrized by the **Segre variety**:
$\mathrm{Seg}(\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n) :=$

$$\begin{aligned} &\{T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \mid T = a \otimes b \otimes c, \text{ for some } a, b, c \in \mathbb{C}^n\} \\ =&\{T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \mid \mathrm{rk}(T) = 1\}. \end{aligned}$$

The Segre variety is the image of the map

$$\begin{aligned} \mathrm{Seg} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n &\longrightarrow \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \cong \mathbb{C}^{n^3} \\ (u, v, w) &\mapsto (u_1 v_1 w_1, \ldots, u_i v_j w_k, \ldots, u_n v_n w_n) \\ (u, v, w) &\mapsto u \otimes v \otimes w. \end{aligned}$$

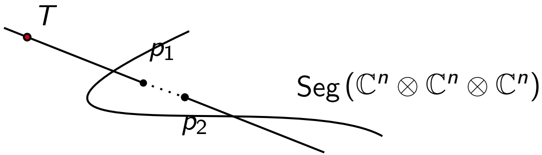
What happens for rank-2 tensors?

A **rank-2 tensor** $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is a tensor that can be written as the sum of 2 rank one tensors $T = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_2$.

What happens for rank-2 tensors?

A **rank-2 tensor** $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is a tensor that can be written as the sum of 2 rank one tensors $T = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_2$.

The **geometric picture** to have in mind for $T = p_1 + p_2 = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_2$ is

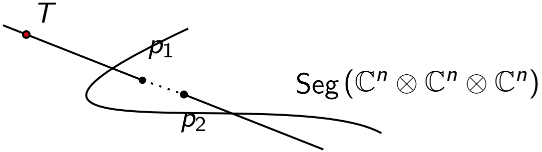


$T \in \langle p_1, p_2 \rangle$ lives on a **secant line**.

The variety of secant lines

The **variety of secant lines** of a Segre helps you understand rank-2 tensors:

honest secant lines = rank-2 tensors



tangent lines = rank-3 tensors

points of the Segre = rank-1 tensors

These are all things that I learned during my PhD here in Trento...

Why are tensors **sooooo** important?

A tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ of rank r is **identifiable** if it can be decomposed in a **unique way** as

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i.$$

Why are tensors **sooooo** important?

A tensor $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ of rank r is **identifiable** if it can be decomposed in a **unique way** as

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i.$$

Having a unique decomposition allows to reconstruct the initial parameters of the model you are analyizing!

Home > Mediterranean Journal of Mathematics > Article

Identifiability of Rank-3 Tensors

Open access | Published: 09 July 2021

Volume 18, article number 174, (2021) Cite this article

Mediterra

Aims and sco

Submit manu

Download PDF

You have full access to this open access article

Edoardo Ballico, Alessandra Bernardi & Pierpaola Santarsiero

Use our pre-

Home > Bollettino dell'Unione Matematica Italiana > Article

An algorithm for the non-identifiability of rank-3 tensors

Open access | Published: 15 March 2023

Volume 16, pages 595–624, (2023) Cite this article

Bollettino d

Italiana

Aims and scope

Submit manus

Download PDF

You have full access to this open access article

Pierpaola Santarsiero

Use our pre-su

Then I moved to Leipzig...

MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

INSTITUTE RESEARCH NE

/ NONLINEAR ALGEBRA PEOPLE - 2022 /

Pierpaola Santarsiero

September 2, 2022

My name is Pierpaola Santarsiero and I joined the Nonlinear Algebra group at the MPI MiS as a postdoctoral researcher in March 2022. I recently finished my PhD at the University of Trento, Italy, under the supervision of Alessandra Bernardi.

My research interests are in multilinear algebra and algebraic geometry. During my PhD I worked on the identifiability problem for tensors and I looked at tensors from the geometric perspective given by secant varieties of Segre varieties and 0-dimensional schemes. At the moment I'm exploring different perspectives on tensors as well as learning new exciting math.

In my free time here in Leipzig I desperately try to avoid cucumbers, but even though *Gurke* is my enemy I still can stand it in a very well prepared gin tonic.

… and I started to learn some new math!

random and **metric** algebraic geometry

Home > La Matematica > Article

Average Degree of the Essential Variety

Original Research Article | Open access | Published: 10 May 2024

Volume 3, pages 753–776, (2024) Cite this article

La Matemat

Aims and scop

Submit manu

Download PDF

You have full access to this open access article

Paul Breiding, Samantha Fairchild, Pierpaola Santarsiero & Elima Shehu

Degree of the subspace variety

Paul Breiding, Pierpaola Santarsiero

Subspace varieties are algebraic varieties whose elements are tensors with bounded multilinear rank. In this paper, we compute their degrees by computing their volumes.

Comments: 14 pages, comments are welcome!

- TeX Sourc
- Other Forr

view license

Current browse
math.AG

< prev |

new | recent | 2

Change to brow
math

Symmetric Geometric rank

With J. Lindberg

Let $T \in \mathrm{Sym}^3\mathbb{C}^n$ and call $A_1, \ldots, A_n$ the slices of T.

Symmetric Geometric rank

With J. Lindberg

Let $T \in \mathrm{Sym}^3\mathbb{C}^n$ and call $A_1, \ldots, A_n$ the slices of T.
The **symmetric geometric rank** of T is

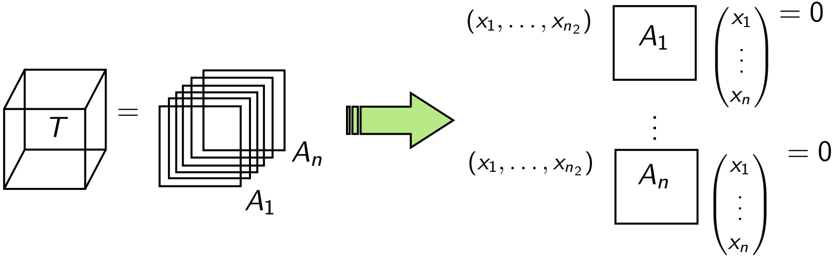
$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}$$

Symmetric Geometric rank

With J. Lindberg

Let $T \in \mathrm{Sym}^3\mathbb{C}^n$ and call $A_1, \ldots, A_n$ the slices of T.
The **symmetric geometric rank** of T is

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}$$



Symmetric Geometric rank

With J. Lindberg

$F \in \mathbb{C}[x_1, \ldots, x_n]_{(3)} \leftrightarrow T \in \mathrm{Sym}^3\mathbb{C}^n$,
$A_1, \ldots, A_n$ the slices of T.

$$\begin{aligned}\mathrm{SGR}(T) &:= \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\} \\ &= \mathrm{codim}\left\{x \in \mathbb{C}^n \mid \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.\end{aligned}$$

Symmetric Geometric rank

With J. Lindberg

$F \in \mathbb{C}[x_1, \ldots, x_n]_{(3)} \leftrightarrow T \in \mathrm{Sym}^3\mathbb{C}^n$,
$A_1, \ldots, A_n$ the slices of T.

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}$$
$$= \mathrm{codim}\left\{x \in \mathbb{C}^n \mid \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.$$

Recall:

- The zero locus $X_F = \{F = 0\} \subset \mathbb{C}^n$ of F is an hypersurface.

Symmetric Geometric rank

With J. Lindberg

$F \in \mathbb{C}[x_1, \dots, x_n]_{(3)} \leftrightarrow T \in \mathrm{Sym}^3\mathbb{C}^n$,
$A_1, \dots, A_n$ the slices of T.

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}$$
$$= \mathrm{codim}\left\{x \in \mathbb{C}^n \mid \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.$$

Recall:

- The zero locus $X_F = \{F = 0\} \subset \mathbb{C}^n$ of F is an hypersurface.
- A point $p \in \mathbb{C}^n$ is **singular** for X_F if $F(p) = 0$ and $\frac{\mathrm{d}F(p)}{\mathrm{d}x_i} = 0$ for all i.

Symmetric Geometric rank

With J. Lindberg

$F \in \mathbb{C}[x_1, \ldots, x_n]_{(3)} \leftrightarrow T \in \mathrm{Sym}^3\mathbb{C}^n$,
$A_1, \ldots, A_n$ the slices of T.

$$\mathrm{SGR}(T) := \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\}$$
$$= \mathrm{codim}\left\{x \in \mathbb{C}^n \mid \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.$$

Recall:

- The zero locus $X_F = \{F = 0\} \subset \mathbb{C}^n$ of F is an hypersurface.
- A point $p \in \mathbb{C}^n$ is **singular** for X_F if $F(p) = 0$ and $\frac{\mathrm{d}F(p)}{\mathrm{d}x_i} = 0$ for all i.
- The singular locus of X_F is $\mathrm{Sing}(F) = \{\frac{\mathrm{d}F}{\mathrm{d}x_0} = \cdots = \frac{\mathrm{d}F}{\mathrm{d}x_n} = 0\}$.

Symmetric Geometric rank

With J. Lindberg

$F \in \mathbb{C}[x_1, \ldots, x_n]_{(3)} \leftrightarrow T \in \mathrm{Sym}^3\mathbb{C}^n$,
$A_1, \ldots, A_n$ the slices of T.

$$\begin{aligned}\mathrm{SGR}(T) &:= \mathrm{codim}\{x \in \mathbb{C}^n \mid x^T A_1 x = \cdots = x^T A_n x = 0\} \\ &= \mathrm{codim}\left\{x \in \mathbb{C}^n \mid \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\}.\end{aligned}$$

$$\mathrm{SGR}(T) = \mathrm{codim}_{\mathbb{C}^n}(\mathrm{Sing}(F)).$$

SGR is useful to solve problems in extremal combinatorics

An *undirected uniform hypergraph* is a couple (V, E), where

- $V = \{1, \dots, n\}$, vertices
- $E \subset 2^V$ contains the edges of the hypergaph and all of them have cardinality 3.

Starting from (V, E) we can always construct a tensor

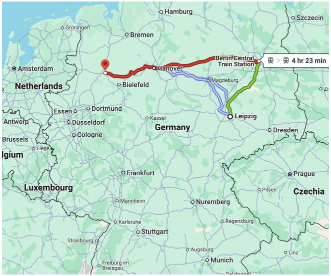
$$T = (t_{i,j,k}) \text{ where } \begin{cases} t_{i,j,k} = 1, & \text{if } \{i, j, k\} \in E \\ t_{i,j,k} = 0, & \text{otherwise.} \end{cases}$$

The independence number of an hypergraph is

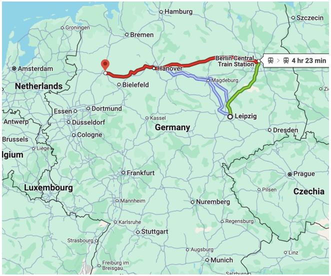
$$\# \text{ largest subset of } V \text{ containing no edges of } (V, E).$$

This number is **difficult to compute** in general, and SGR gives an upper bound of it.

Then I moved to Osnabrueck



Then I moved to Osnabrueck



And I started working more on tensors coming from applications

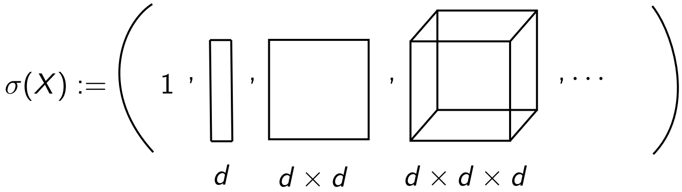
Signature tensors

[Chen $\sim$ 50s]

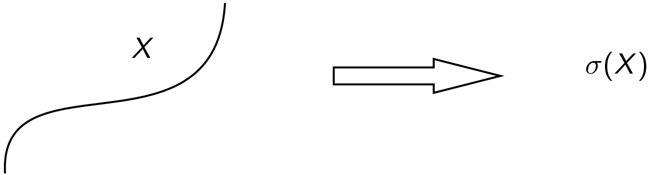
Consider a **good** path

$$X : [0,1] \longrightarrow \mathbb{R}^d$$
$$t \mapsto (X_1(t), \ldots, X_d(t)).$$

The **signature** of X is a sequence $\sigma(X) = \left(\sigma^{(k)}(X)\right)_k$ where $\sigma^{(k)}(X) \in (\mathbb{R}^d)^{\otimes k}$ with the convention that $\sigma^{(0)}(X) = 1$:



Signature encodes information of the path



Theorem (Chen, '58)

$X \cong Y$ if and only if $\sigma^{(k)}(X) = \sigma^{(k)}(Y)$ for all k.

- Journal of Pure and Applied Algebra
Volume 229, Issue 1, January 2025, 107807

  ## Decomposing tensor spaces via path signatures

  Carlos Améndola [a], Francesco Galuppi [b], Ángel David Ríos Ortiz [c], Pierpaola Santarsiero [d], Tim Seynnaeve [e]

  Show more

- [Submitted on 29 Jul 2024]

  ## Rank and symmetries of signature tensors

  Francesco Galuppi, Pierpaola Santarsiero

  The signature of a path is a sequence of tensors which allows to uniquely reconstruct the path. In this paper we propose a systematic study of basic properties of signature tensors, starting from their rank, symmetries and conciseness. We prove a sharp upper bound on the rank of signature tensors of piecewise linear paths. We show that there are no skew-symmetric signature tensors of order three or more, and we also prove that specific instances of partial symmetry can only happen for tensors of order three. Finally, we give a simple geometric characterization of paths whose signature tensors are not concise.

  Subjects: **Algebraic Geometry (math.AG)**
MSC classes: 14N07, 15A69, 15A72, 60L10

- detecting the shape of a path from its signatures (work in progress)

and since April 2024 I’m back in Italy and I’m part of the INABAG group

INABAG

Italian Network for Applied and Birational Algebraic Geometry

INABAG

It is with great sadness that we announce the loss of our friend and colleague **Gianfranco Casnati.** Gianfranco passed away unexpectedly on November 29th 2023, and will be sorely missed by all of us.

We are a group of geometers and algebraists based in Italy, whose common research interests lie within the theory of tensors from the commutative algebraic, the birational algebro-geometric, and the applied viewpoints. Our research team cultivates a collaborative atmosphere by means of joint seminars and collective activities (see the "Activities" tab) that foster the growth of its members. We also have a long track record of joint publications.

Our research is suported by the Ministero dell'Università e della Ricerca (Ministry of University and Research) through the award of four research grants within the scheme PRIN 2022. We have been recruiting a number of postdocs (see the "Job opportunities" tab for info), who naturally blended in our team and took part in our activities.

The PRIN2022 Projects and their units

Kick-off Meeting of the INABAG Research Group,
featuring talks by the postdocs hired within three PRIN grants.

Family photo

Thank you for the attention!